Mathematical Result Paper

Conjecture and Proof

Sarah Amstutz

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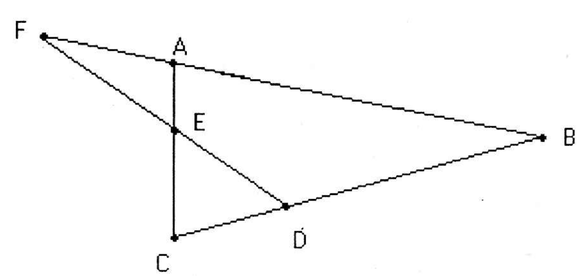
Menelaus Theorem

I chose to present the Menelaus Theorem, finding an interesting reference on the Internet about a letter from Albert Einstein to his friend Max Wertheimer. Apparently they were discussing the beauty of proofs, and Einstein used two proofs of the Menelaus Theorem, one ugly, and the other elegant, (in his opinion) to illustrate his point. I will discuss and analyze these proofs in this paper.

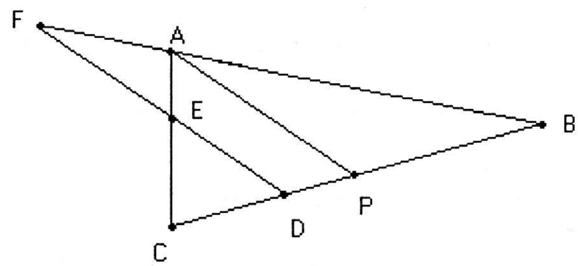
Einstein wrote, (of a proof) "...we are completely satisfied only if we feel of each intermediate concept that it has to do with the proposition to be proved." The Menelaus Theorem states:

Let three points F, D, and E, lie respectively on the sides AB, BC, and AC of ▲ABC. If the points are collinear, then AF/BF * BD/CD * CE/AE = 1.



As you can see, the theorem allows, and requires, that at least one of the points lie on the extension of the corresponding side, since a straight line cannot cross all three sides of a triangle. (On an interesting side note, in a 1945 Russian math competition, one boy did not see the fact that a straight line can't cross a three sides of a triangle as obvious. For that insight, he was awarded first prize, even though he didn't solve a single problem.) This "If, then" statement requires a direct proof.

Proof 1.



Suppose ABC is a triangle. Suppose F, D, and E are points lying on sides AB, BC, and AC of the triangle ABC, respectively. Suppose F, D, and E are collinear. Consider AP, parallel to DE, with P lying on side BC of ▲ABC. Since AP and DE are parallel, triangles ABP and BDF are similar, as are triangles ACP and CDE. Since ▲ABP and ▲BDF are similar, AF/BF = PD/BD. Since ▲ACP and ▲CDE are similar, AE/CE = PD/CD. Then AF/BF * BD/PD = 1 and AE/CE * CD/PD = 1. Also, (AF/BF * BD/PD) * (CE/AE * PD/CD) = 1. So, AF/BF * (BD * PD / PD * CD) * CE/AE = 1, and thus, AF/BF * BD/CD * CE/AE = 1.

How would this proof work if all 3 of D, E, F lie on extensions of the si

Einstein states of this proof, "Although the first proof is somewhat simpler, it is not satisfying. For it uses an auxiliary line which has nothing to do with the content of the proposition to be proved, and the proof favors, for no reason, the vertex A, although the proposition is symmetrical in relation to A, B, and C. The second proof, however, is symmetrical, and can be read off directly from the figure." Many times in this course, we have felt the dissatisfaction and discontent

Einstein speaks of from an object that seems to have appeared from nowhere, especially in construction proofs.

Proof 2.

Suppose ABC is a triangle. Suppose F, D, and E are points lying on sides AB, BC, and AC of the triangle ABC respectively. Suppose F, D, and E are collinear. Since the area of a triangle can be written as half the product of two sides and the sine of the angle in between, the area of ▲AEF = ½ * AF * FE * sin (angle F). In addition, the area of ▲BDF = ½ * BF * FD * sin (angle F). Dividing these two equations gives area(▲AEF)/area(▲BDF) = AF * EF / BF * DF. Consequently, with pairing up triangles by common sine-of-angle values (notice that angles DEC and AEF are equal and the sines of angles BDF and EDC are equal since sin(angle) = sin(180 – angle)), area(▲CDE)/area(▲AEF) = ED * CE / AE * FE, and area(▲BDF)/area(▲CDE) = BD * DF / CD * ED. Multiplying these last three equations, they simplify to 1 = AF * BD * CE / BF * CD * AE. Thus AF/BF * BD/CD * CE/AE = 1.

This second proof is a bit more complex, yet more satisfying, at least to Einstein. Of course, "Elegance, as beauty, is in the eye of the beholder," as Alex Bogomolny states in his web site at www.cut-the-knot.com. A person such as Bogomolny, who doesn't mind "artificial" constructs, might appreciate the simpler proof a bit more.

Reference: A. Bogomolny, *The Menelaus Theorem* and *Menelaus Theorem: Proofs Ugly and Elegant, A. Einstein's View*, www.cut-the-knot.com, 1996-2000