COOPERATIVE GAMES ON WEIGHTED GRAPHS

by

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Abstract: The class of weighted graph cooperative games is described and it is shown that the Shapley value, tau value, and nucleolus give identical allocations on these games. A more general sufficient condition is stated for these three allocation methods to yield identical allocations.

Introduction

An n-person cooperative game is a pair (N, v) where $N = \{1, 2, ..., n\}$ is a set of players and where v is a real-valued function on the set of all subsets of N, where $v(\emptyset) = 0$. We interpret v(S) as the value obtainable by the coalition S =when the members of S work together. A game is superadditive if for all coalitions S and T where $S \cap T = \emptyset$, $v(S \cup T) \geq v(S) + v(T)$. We will use S^c to denote the complement of S: N\S.

Several special classes of games, defined with respect to a weighted graph or network, have been considered in the literature. Bird [1], Megiddo [6], Granot and Huberman [3], Rosenthal [11], and others have studied minimum cost spanning tree games where the players correspond to the regular vertices of a weighted graph and v(S) is the negative of the minimum weight tree that spans S and a special vertex. Kalai and Zemel [4], Dubey and Shapley[2], and others have studied network flow games where players control subsets of edges in a network and v(S) is the maximum flow through the subnetwork induced by the edges controlled by players in S. Shapley and Shubik [14] and others have studied the assignment game where players are the vertices of a weighted bipartite graph and v(S) is the maximum weight matching on the subgraph induced by S. Potters, Curiel and Tijs [10] have studied traveling salesman games where the players correspond to the regular vertices of a weighted graph and v(S) is the negative of the minimum weight circuit covering S and the special vertex. Myerson [7], Owen [9] and Rosenthal [11] considered games in which cooperation is limited by a "communication" graph. In this paper, we study games in which the players are the vertices of a weighted graph and v(S) is the sum of the weights on the edges of the subgraph induced by S.

A vector $\mathbf{x} = (x_1, x_2, ..., x_n)$ with real components is an imputation for the game if $x_i \geq v(i)$ for all i contained in N (individual rationality), and

$$\sum_{i=1}^{n} x_i = v(N) \quad (efficiency).$$

An allocation method is a function from games to imputations. In this paper we consider three of the most widely known allocation methods.

The Shapley value ~~See~~ [13] measures each player's marginal contribution to the grand coalition, and averages these values over all possible permutations of the players. Formally, the Shapley value for player i is defined as

$$\phi_i = \sum_{S \ni i} \frac{(s-1)!(n-s)!}{n!}[v(S) - v(S - \{i\})].$$

Define the marginal vector $M_i = v(N) - v(N\backslash\{i\})$, which is the amount player i will contribute by joining to form the grand coalition. This is considered as the most that player i can hope to receive if working cooperatively with the other players. The remainder for player i is calculated:

$$R(S, i) = v(S) - \sum_{j \in S-\{i\}} M_i.$$

From this a lower bound for an allocated amount for player i is found. The minimal right for player i is $m_i = \min \{R(S, i): i \in S \subseteq N\}$. The τ-value ~~See~~[15] is the unique efficient vector lying on the line determined by m(x) and M(x).

Define the excess of coalition S with respect to the imputation x as

$$e(x,S) = v(S) - \sum_{i \in S} x_i.$$

This value represents the "size of the complaint" the coalition S would have against the allocation x. Define the excess vector of the imputation x as $e(x) = [e(x,S_1), e(x,S_2), ..., e(x,S_{2^n})]$, where $e(x,S_i) \geq e(x,S_{i+1})$. To define a lexicographic ordering on x and y, we say $x < y$ if there exists an i such that for $j = 1, ..., i$, $x_j = y_j$, and $x_{i+1} < y_{i+1}$. The nucleolus ~~See~~[12] is the imputation which minimizes e(x) lexicographically. It is in one sense the "middle" of the core of a game, if the core is nonempty. The nucleolus can be found with a sequence of linear programs.

We now present Kohlberg's characterization of the nucleolus. Let $\mathcal{B} = \{S_1, S_2, ..., S_m\}$ be a collection of subsets of N. $\mathcal{B}$ is balanced if we can find a balancing vector $y = (y_1, ..., y_m)$ such that, for every player i,

$$\sum_{j:\, i \in S_j} y_j = 1,$$

and all $y_j > 0$. Given an imputation x, let $\mathcal{B}_k$ be the set of all coalitions with k^{th} maximal excess, that is, $e(x,S) = \alpha_k$ for all $S \in \mathcal{B}_k$ and $\alpha_k > \alpha_{k+1}$. Let

$$C_k = \bigcup_{i=1}^{k} \mathcal{B}_i .$$

We call the $\mathcal{B}_k$'s and the $\mathcal{C}_k$'s the arrays determined by x.

Theorem 1: [5] The imputation **x** is the nucleolus of a superadditive game (N, v) if and only if the array $C_1, ..., C_q$ determined by **x** consists of only balanced collections.

Weighted Graph Games

Weighted graph games are a class of games in which the value of all coalitions with more than two players depends on the values of the two-player coalitions. Given a set of players N = {1, 2, ..., n} , define the characteristic function as

$$v(S) = \begin{cases} 0 & , if|S|<2 \\ w_S & , if|S|=2 \\ \sum_{R\subset S} w_R & , if|S|>2 \end{cases}$$

A way to visualize this is to draw a graph G = (N, E), where each player is represented by an element of the vertex set N, and the value of the coalition (ij) is represented by $w_{(ij)}$, the weighted edge of E with endpoints i and j. Note that we define $w_{(ij)} = 0$ if $(ij) \notin E$. On a coalition $S \subseteq N$ define

(1) an interior edge of S as an edge in E with both vertices in S;

(2) an exterior edge of S as an edge with no vertices in S; and

(3) a boundary edge of S as an edge with one vertex in S.

The value of a coalition S is defined as the sum of the weights of the interior edges of S.

Theorem 2: On weighted graph games, the nucleolus, the Shapley value and the tau value are identical: $\nu_i = \tau_i = \phi_i = 1/2$ * (the sum of the weights of the edges adjacent to i).

Proof for nucleolus: Let **x** be defined by $x_i = 1/2$ * (the sum of the weights of the edges adjacent to i). We will use Kohlberg's theorem to prove that **x** is the nucleolus. Now for each $S \subset N$,

$$e(x, S) = v(S) - \sum_{i \in S} x_i$$

= (the sum of the weights of the interior edges of S) - 1/2*($\sum_{i \in S}$ the sum of the weights of the edges adjacent to i)

= (the sum of the weights of the interior edges of S) - 1/2 * (2 * the sum of the weights of the interior edges of S + the sum of the weights of the boundary edges of S)

= -1/2 * (the sum of the weights of the boundary edges of S).

It also follows that $e(x, S^c) = -1/2$ * (the sum of the weights of the boundary edges of S^c). Since the boundary edges of S are the boundary edges of S^c, $e(x, S) = e(x, S^c)$. Thus, for every $S \subset \mathfrak{B}_i$, $S^c \subset \mathfrak{B}_i$. Since each player is in the same number of coalitions in each $\mathfrak{B}_i$, each $\mathfrak{B}_i$ is balanced and so each $\mathfrak{C}_i$ is balanced [8 , p 158]. Now by Theorem 1, **x** is the nucleolus.

Proof for Shapley value: Consider an edge e with vertices i and j. When calculating the Shapley value the only player that will receive the weight of edge e is player i or player j, whichever one comes later in the player permutation. Players i and j appear after each other an equal number of times so they will split the value of edge e, each receiving half of the weight of edge e. So the Shapley value for player i is 1/2 * (the sum of the weights of the edges adjacent to vertex i).

Proof for τ-value: To find the τ-value for player i, first we find the marginal vector M corresponding to v with

$$M_i(v) = v(N) - v(N - \{i\})$$

= (the sum of the weights of all edges of the graph) -
(the sum of the weights of the edges which are not adjacent to i)
= the sum of the weights of the edges adjacent to i.

Next the remainder for player i in the coalition S is calculated with

$$R(S,i) = v(S) - \sum_{j \in S} M_j$$

= (the sum of the weights of the interior edges of S) - (the sum of the weights of the edges adjacent to j for each $j \in (S - \{i\})$).

Set $m_i = \max \{R(S, i): i \in S \subset N\}$. This occurs when S is a singleton since for each set with more than one member, the sum of the weights of the edges adjacent to j for each $j \in (S - \{i\})$ is at least as large as the sum of the weights of the interior edges of S. So $m_i = 0$ for all $i \in N$. Thus,

$$\tau_i = (1-\alpha)m_i + \alpha M_i$$

$$\tau_i = \alpha * \text{(the sum of the weights of the edges adjacent to i)},$$

for the unique value of α which makes τ efficient. Clearly, $\tau_i(v) = 1/2 *$ (the sum of the weights of the edges adjacent to i).

A Sufficient Condition for $\phi = \tau = \nu$

Theorem 1 requires that the $\mathcal{C}_k$'s determined by the nucleolus be balanced. The nucleolus of weighted graph games satisfies a much stronger property: the $\mathcal{B}_k$'s are closed under complementation, that is, $S^c \in \mathcal{B}_k$ for all $S \in \mathcal{B}_k$. This property is a sufficient condition for the three allocation methods to yield identical results.

Theorem 3: Suppose (N, v) is a superadditive game, x is an imputation, and $\mathcal{B}_1, ..., \mathcal{B}_q$ is the array determined by x. If $\mathcal{B}_k$ is closed under complementation for all $k \in \{1, ..., q\}$, then $\phi = \tau = \nu = x$.

Proof: That n = x follows directly from Theorem 1. Now the complementation property implies that for all $S \subset N$, $e(x, S) = e(x, S^c)$. (1)

$$v(S) - \sum_{i \in S} x_i = v(S^c) - \sum_{j \in S^c} x_j.$$

If |N| is even, then by definition

$$\phi_i = \sum_{S \ni i} \frac{(s-1)!(n-s)!}{n!}[v(S) - v(S - \{i\})]$$

$$= \sum_{S \ni i, |S| \leq \frac{n}{2}} \frac{(s-1)!(n-s)!}{n!}\left[v(S) - v(S - \{i\}) + v((S - \{i\})^c) - v(S^c)\right]$$

$$= \sum_{S \ni i, |S| \leq \frac{n}{2}} \frac{(s-1)!(n-s)!}{n!}\left[\sum_{j \in S} x_j - \sum_{j \in (S - \{i\})} x_j + \sum_{j \in (S - \{i\})^c} x_j - \sum_{j \in S^c} x_j\right]$$

$$= 2v_i \sum_{S \ni i, |S| \leq \frac{n}{2}} \frac{(s-1)!(n-s)!}{n!}$$

$$= 2x_i \sum_{s=1}^{\frac{n}{2}} \frac{(n-1)!}{(s-1)!(n-s)!} * \frac{(s-1)!(n-s)!}{n!}.$$

$$= 2x_i \sum_{s=1}^{\frac{n}{2}} \frac{1}{n}.$$

$= x_i$

If |N| is odd, a similar argument shows that $x_i = \phi_i$.

Now we consider the τ-value. First,

$$M_i = v(N) - v(N - \{i\})$$
$$= x_i - e(x, N-\{i\})$$
$$= x_i - e(x, \{i\})$$

$$= 2x_i - v(\{i\})$$

Where the third equality follows from (1) Second,

$$\begin{aligned} R(S, i) &= v(s) - \sum_{j \in S\text{-}\{i\}} M_j \\ &= e(x, S) + x_i + \sum_{j \in S\text{-}\{i\}} e(x,\{j\}) \\ &= e(x, S^c) + x_i + \sum_{j \in S\text{-}\{i\}} e(x,\{j\}) \\ &= v(S^c) + \sum_{j \in S\text{-}\{i\}} v(\{j\}) - v(N) + 2x_i \\ &\leq v(N\text{-}\{i\}) - v(N) + 2x_i \\ &= R(\{i\}, i) \end{aligned}$$

where the third equality follows from (1) and the inequality follows from superadditivity. Hence,

$$\begin{aligned} m_i &= \max \{R(S,i) : i \in S \subseteq N\} \\ &= R(\{i\}, i) \\ &= v(\{i\}) \end{aligned}$$

The unique efficient linear combination of M and m is given by,

$$\tau_i = \frac{1}{2}M_i + \frac{1}{2}m_i = x_i$$

The class of completely symmetric games ($v(S) = v(T)$, if $|S| = |T|$) also has the property that the three allocation methods considered are identical. This does not cover all possible games in which the three allocation methods considered are identical, as shown in the following example.

Example 1: Let $\mathcal{B}_5 = \{ \{3, 4, 5\}, \{1, 2, 5\}, \{2, 3, 4\}, \{1, 4, 5\}, \{1, 2, 3\}\}$ and

$\mathcal{B}_4 = \{S \subseteq N: |S| = 3 \text{ and } S \notin \mathcal{B}_5\}$. Define $N=\{1, 2, 3, 4, 5\}$ and

$$v(S) = \begin{cases} 0 \ , \textit{if } |S| = 1 \textit{ or } 2 \\ 1 \ , \textit{if } S \in \mathcal{B}_5 \\ 2 \ , \textit{if } S \in \mathcal{B}_4 \\ 9 \ , \textit{if } |S| = 4 \\ 15 \ , \textit{if } S = N \end{cases}$$

It is easy to verify that $\phi = \tau = \nu = (3, 3, 3, 3, 3)$.

This prompts the following question: For what class of superadditive games are the

Shapley value, tau-value and the nucleolus identical? We know that it must include the completely symmetric games and the games described in Theorem 2. A natural class of games to consider is the class of games in which the $\mathcal{B}_k$'s determined by the nucleolus are balanced. This class includes completely symmetric games and the games described in Theorem 2. Note that Example 1 is not in this class. We conclude with an example in this class for which the three allocation methods yield different allocations.

Example 2: Define N = { 1, 2, 3, 4}, and v(S) = 0 if |S| =1,

v({1, 2}) = v({1, 3}) = 3, v({1, 4}) = 1, v({2, 3}) = 2, v({2, 4}) = v{3, 4}) = 5

v(1, 2, 3}) = v({2, 3, 4}) =8, v({1, 2, 4}) = v({1, 3, 4}) = 5, v(N) = 13

It is easy to verify that ϕ = (13/6, 4, 4, 17/6) and τ = (5/2, 4, 4, 5/2). Furthermore, ν = (2,4,4,3) and the array determined by ; ν is given by $\mathcal{B}_1$ = { Ø, N}, $\mathcal{B}_2$ = { {1}, {2,4}, {3,4}, {1,2,3}}, $\mathcal{B}_3$ = { {4}, {1,2}, {1,3}, {2,3,4}} and $\mathcal{B}_4$ contains all other coalitions. Clearly, each $\mathcal{B}_k$ is balanced.

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