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Cooperative Games: Cliques in Graphs

Let G be a simple graph on n vertices, V(G) the set of vertices of G. Define a worth function f: $S \subseteq V(G) \rightarrow Z$ by f(S) = −m(S), where m(S) = size of largest clique contained in the subgraph generated by the points of S. Since m(S+T) = n implies S+T contains a clique of order n, and dividing the vertex set of a complete graph of order n into two parts produces two complete subgraphs whose orders add to n, $m(S) + m(T) \geq m(S+T)$ for any set S, T of vertices (not necessarily of a complete graph) and hence $f(S+T) \geq f(S) + f(T)$. So f is a superadditive function, and (V(S), f) is a superadditive cooperative game.

1. The Shapley Value

Calculating the Shapley value for an arbitrary graph G is decidedly nontrivial. At this time solutions exist only for complete graphs, complete n-partite graphs, and 2-partite graphs with partition sets of size k which are regular of degree 1 or n−1.

1.

a. Complete graphs.

Let G be a complete graph, and S a subgraph of G on k vertices. Then S is complete, and furthermore S−{x} is a complete subgraph of order |S|−1, so the Shapley value is simply $-n!/n! = -1$.

b. Complete n-partite graphs.

Begin by considering the bipartite case. Let G be a complete bipartite graph of order k, and let the sets R, S of order r, s respectively be the bipartite partition of the vertices of G. Claim: for x in R, the Shapley value of x is 1/r.

Proof:

The Shapley value of x is the average of the marginal contributions of x over all orderings of V(G). A given vertex contributes at most 1 to the clique size, so x contributes either 0 or -1 to f(S) for any given ordering. G is bipartite, so the maximal clique size is 2; hence, x contributes 1 to the maximal clique size of a given ordering only if x is the 1st vertex (creates a clique of size 1) or x creates the first edge (clique of size 2). This turns out to correspond precisely to the case in which x is the first element of R to occur in the ordering. For, any vertex in R and any vertex in S generate one edge; hence, if the first vertex lies in R, the first edge will be generated by a vertex in S. So if the first vertex lies in R, it must be x. If the first vertex lies in S, then the first point of R will make an edge, so it must be x. Either way, x is the first point in R. The ratio of the number of times x contributes 1 to the clique size to the total possible orderings (the Shapley value for x) is therefore exactly equivalent to the probability that x occurs first in R, which is 1/r.

Now, consider the case of an n-partite graph. The graph has maximal clique size n (selecting one point from each of the n partition sets generates K_n). Furthermore, for any given ordering, elements in a set S_i contribute exactly 1 to the clique size, since adding an element from a new set generates K_{m+1} from K_m if m sets are already represented. Since no two elements of S_i share an edge, no larger cliques can be generated. Hence, an element x contributes 1 to the clique size iff it is the first element of its set to occur. By analogy to the bipartite case, the probability of x contributing 1 to the clique size is therefore simply $1/S_i$.

c. 2-partite graphs with partition sets of order k, regular of degree n-1 or 1; connected 2-partite graphs regular of degree 2.

All three cases are trivial; by symmetry, the Shapley value for all three is simply 1/k. (For 2-partite graphs with partition sets of order k, regular of degree n-1, arrange the graph such that the k pairs of unconnected points face each other; then the graph is clearly symmetric. Connected bipartite graphs regular of degree 2 are simply cycles and therefore symmetric.)

2. Banzhaf Values.

a. Complete graphs.

For x a vertex in a complete graph, S a subset of V(G) containing x and of order k, $f(x)=(-1)$; $f(S) = -k = -(k-1) +$

$(-1) = f(S-\{x\}) + S(\{x\})$, so $f(S) - f(S-\{x\}) - f(x) \equiv 0$ and the Banzhaf value is simply $f(x) = -1$.

b. Complete n-partite graphs.

Define S, S_j as above.

For $x \in S_j$ in a given coalition S, $f(S-\{x\}) = f(S)$ iff there is an additional element of S_j in S. This is clear since if x completes a clique of order m, $K_m - x + x'$, $x' \in S_j$, is also a clique of order m since the graph is complete n-partite. Hence $f(S)-f(S-\{x\})-f(x) = 0-(-1) = 1$ iff $x, x' \in S \cap S_j$. For x the only element of S_j in S, $f(S-\{x\}) = f(S)+1$; $f(x) = -1$, so $f(S) - f(S-\{x\}) - f(x) = f(S) - f(S-\{x\}) - f(x) = -1 - (-1) = 0$.

Now, for a coalition S of order s, there are

$$\sum \binom{m_j}{i}\binom{m-m_j}{s-i-1} \text{ ways to pick } x,\ i \text{ points from } S_j,$$

and $s-i-1$ points from $V(G)-S_j$.

Hence the Banzhaf value for $x \in S_j$ is

$$f(x) + \frac{\sum_{S \ni x}[f(S)-f(S-\{x\})-f(x)]}{\sum_{y \in V(G)}\sum_{S \ni y}[f(S)-f(S-\{y\})-f(y)]}\left[f(V(G)) - \sum_{y \in V(G)} f(y)\right]$$

$$= -1 + \frac{\sum \binom{m_j}{i}\binom{m-m_j}{s-i-1}}{\sum\sum \binom{m_k}{i}\binom{m-m_k}{s-i-1}}[-n+m] = -1 + \frac{(2^{m_j-1}-1)2^{m-m_j}}{\sum_{k=1}^{n} m_k(2^{m_k-1}-1)2^{m-m_k}}[-n+m]$$

There are $(2^{m_j-1}-1)2^{m-m_j}$ coalitions that contain x, at least one other element of S_j and some subsets of $V(G)-S_j$.

3. Nucleolus for graphs G consisting of two cycles of length r and length s, respectively. $(r,s \geq 4)$

Claim: for r, s of same parity, the nucleolus assigns value

2/(r+s) to each point in G; for r odd, s even, each point in R has value 4/(2r+s), each point in S has value 2/(2r+s).

Proof:

Let v(S) be the worth of a coalition S, x′(S) = sum of v(a), a in S, x(S) = −x′(S).

min max x $S \neq \phi, N$

Want to minimize [the] maximum [coalitional] excess, i.e. minimize v(S)−x(S). This is equivalent to minimizing −(x′(S) − v(S)), so we may flip signs throughout and minimize x(S) − m(S).

Now, since all points in cycle R are equivalent, and all points in S are equivalent, the nucleolus must assign the same value x to all points in R, and the same value y to all points in S. Efficiency gives

(1) $rx + sy = 2$.

For m(T) = 2, x(T) − m(T) is ~~min~~ [max]imized when have all but one point in the set T; thus, for m(T) = 2,

(2) $\alpha \geq (r-1)x + sy - 2$

(3) $\alpha \geq rx + (s-1)y - 2$.

For m(T) = 1, the ~~min~~ [max]imal x(T) −m(T) occurs when T is the maximal set containing no edge; clearly, T can be obtained by taking every other point around each cycle. Thus we have

(4) $\alpha \geq \lfloor (r/2) \rfloor x + \lfloor (s/2) \rfloor y - 1$.

(2) and (3), together with (1), are equivalent to

(2′) $\alpha \geq -x$,

(3′) $\alpha \geq -y$. (Note x, y ≥ 0 as are dealing with m(T), not v(T).

Case I. r, s even. Then $\lfloor (s/2) \rfloor = s/2$, $\lfloor (r/2) \rfloor = r/2$ and (4) and (1) yield $\alpha \geq 0$, which is the most restrictive condition since $x, y \geq 0$. So set $\alpha = 0$. This does not affect the case where v(T) = 2, so (2′) and (3′) are unaffected; but the new maximum x(T) for v(T) = 1 is now $((r/2)-1)x + (s/2)y$ or $(r/2)x + ((s/2)-1)y$. This in conjunction with (1) yields only (2′) and (3′), so we get no new information.

Multiply (2′) by r, (3′) by s, and add. This yields $(r+s)\alpha \geq -(rx+sy) = -2$, so that $\alpha \geq -2/(r+s)$. This is the most restrictive condition on α, so equality must hold, i.e. $2/(r+s) = x = y$. Inserting $2/(r+s)$ into equations 1-4 yields equality, so the solution is the nucleolus.

Case II. r odd, s even or vice versa.

Without loss of generality, assume r is odd and s even. Then (4) reduces to (4′) $\alpha \geq -x/2$. Multiplying this inequality by 2r and adding to s(3′) yields $(2r+s)\alpha \geq -2$, so $\alpha \geq -2/(2r+s)$. This is the most restrictive condition on α, so equality holds in (3′) and (4′), so $x = 4/(2r+s)$, $y = 2/(2r+s)$.

Case III. r, s odd. Then (4) reduces to

(5) $\alpha \geq ((r-1)/2)x + ((s-1)/2)y - 1 = -(x+y)/2$.

Multiplying (5) by 2, (2′) by r−1, (3′) by s−1, and adding yields

(6) $(2+m-1+n-1)\alpha \geq -rx - sy = -2$, so that

$\alpha \geq -2/(r+s)$ and, as in Case I, $x = y = 2/(r+s)$.

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Tien-Yee Chiu
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Dear Tien,

I hope that you have been able to relax after eight weeks of mathematics. Unfortunately, I have had administrative work for the Council on Undergraduate Research, lectures for the New Jersey Governor's School, and software library development for Introductory Statistics. Jeanne and I took a few hours off for our anniversary, but now I'm back finishing up REU stuff.

I have enclosed (1) a request for an evaluation of the program, (2) the original of your REU report upon which I have written a number of suggestions, and (3) a copy of participants' whereabouts. Please return your evaluation to me by September 17. If you would like a revised copy of your report to be sent to the NSF and other interested persons, please return your revision by September 17. I have held off sending out copies of student reports pending each student's decision whether or not to revise. There is no requirement to revise your report; it is up to you based upon your time and interest. You have not requested that I send you any of the other reports. If you have changed your mind, just let me know.

Your report is fairly clean except for the Banzhaf value for complete n-partite graphs where it is not clear to me what the indices of summation are. The formula can also be simplified in the manner I have indicted. I think that your results for the complete bipartite and cycle pair graphs are interesting and could form the basis of a short paper. Formulas for the nucleolus of complete bipartite graphs, the Shapley and Banzhaf values for the cycle pair graphs, and the Tau value for both types of graphs are minimum additions. There would also have to be some intuitive "explanation" of the results. For example, why does the Shapley value assign a total of -1 to each set in the bipartite partition but the Banzhaf value does not? Why is parity matching the crucial issue for the nucleolus in the cycle pair graphs? Of course, there are a lot more questions that could be addressed, but I am not sure that they are worth the effort unless we find something interesting in the complete bipartite or two cycle cases.

I will close with a few words about recommendations. I would be happy to write you a recommendation for graduate study or employment upon request. It is my policy to always share with you a copy of my letter of recommendation for you. If there is sufficient time between your request and the receipt deadline, I will send you a first draft for comment. You received two strong but short letters of recommendation when you applied to the REU program. Of course, you will want to stick with mathematicians if applying to graduate school.

Good luck!

Sincerely,

David Housman