The Island of Truthtellers and Liars

Conjecture and Proof
Research Paper
Rebekah Henn
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You have found yourself upon The Island of Truthtellers and Liars.

Assumptions:

1. All questions must be "yes and no" questions and have a single right answer.
2. Islanders know the answer to every "yes or no" question.
3. All islanders are either Truthtellers or Liars.
4. Everyone from the tribe of Truthtellers always tell the truth
5. Everyone from the tribe of Liars always lies.

Definition 1: Subjective Questions are questions whose correct answer changes depending on the person being asked. "Is your name John?" "Are you from the tribe of Truthtellers?" or "Do you like lasagna?" are subjective questions.

Definition 2: Objective Questions are questions that are not subjective (a.k.a. the correct answer does not change with the person you are asking. Questions such as "Is the sky blue?" or "Is Cityopolis to the north?" are objective questions.)

Theorem 3: All questions are either subjective or objective.
Proof: By definition, all questions that are not subjective are necessarily objective. Therefore, all questions are either subjective or objective.

Theorem 4: In answering an objective question a Truthteller and a Liar will always give different answers.
Proof: Suppose you ask any question to two islanders, knowing that one of them is a Truthteller and one is a Liar. Then, because the Truthteller always tells the truth, she will give the correct answer. The Liar, who always lies will give a false answer. The false answer cannot be the same as the correct answer, or it would be the correct answer, and the Liar would have told the truth. Therefore, the Truthteller and the Liar would give different answers.

Theorem 5: If two islander answers an objective question differently then they are from different tribes.
Proof: Suppose two Truthtellers are asked a question. Since they will both give the correct answer, and there can only be one answer, the answers will be the same. They will either both say "yes" or both say "no." If the two islanders are Liars then the first liar will answer the question with the wrong answer. If the correct answer is "yes" that islander would say, "no" and if the correct answer is "no" then she will say "yes." The other liar would do the same. Therefore, both liars would say the same thing either way. Since we know that in answering an objective questions members from different tribes

give different answers, then if one is from the Truthteller tribe and one is from the Liar tribe, then their answers would always differ. Thus, the only way for one islander to say "yes" and the other to say "no" is for the islanders to be of different tribes.

Corollary 6: If two islanders answer an objective question differently, one is necessarily a liar
Proof: Suppose two islanders answer an objective question and give different answers. Then they are necessarily from different tribes. Since there are only two tribes native to the island, then each tribe must be represented. Therefore, one is necessarily a liar.

Corollary 7: If two islanders answer an objective question differently, one is necessarily a truthteller
Proof: Suppose two islanders answer an objective question and give different answers. Then they are necessarily from different tribes. Since there are only two tribes native to the island, then each tribe must be represented. Therefore, one is necessarily a truthteller.

Theorem 8: When you ask two islanders any question, at least one has to tell the truth or at least one has to lie.
Proof: Suppose you ask two islanders a question and neither islander tells the truth. Since they only have to options, then the other two islanders must lie. If two islanders lie, then at least one is lying. So, either at least one has to tell the truth or at least one has to lie.

Theorem 9: When you ask two random islanders any one question, one and only one islander is lying if one and only one islander is telling the truth.
Proof: Suppose one and only one of the islanders tells the truth. The other islander only has two options (telling the truth or lying). Since that islander cannot tell the truth since only one can tell the truth, that islander must be lying. Also, only one islander could then be lying, since we already know the first islander is telling the truth.

Corollary 10: When you ask two random islanders any one question, one and only one islander is telling the truth if one and only one islander is lying.
Proof: Suppose one and only one of the islanders tells a lie. The other islander only has two options (telling the truth or lying). Since that islander cannot lie since only one can lie, that islander must tell the truth. Also, only one islander could then be telling the truth at most, since we already know the first islander is lying.

Theorem 11: If you ask two islanders an objective question and they give the same answer, then that answer cannot be true for one and false for the other.
Proof: Because the question is an objective question, then the correct answer to the question does not change by the person to whom it is asked. Thus if you ask two people the same question, the correct answer will be the same for each person. Since the islanders only answer questions that have only one correct answer, then for both islanders to be correct, they must both say the same thing. Thus, if two islanders give the same answer, the answer must be either true both times or false both times. Thus, if you ask

two islanders an objective question and they give the same answer, then that answer cannot be true for one and false for another.

Corollary 12: If you ask two islanders an objective question and they give the same answer, then they are either both telling the truth or both lying.
Proof: Suppose you ask two islanders an objective question and they give the same answer. There are only four possible truth options. It can be true for one and false for the other, true for the other one and false for the first one, true for both, or false for both. Since it cannot be true for one and false for the other, then that leaves that it is either true for both or false for both. Therefore, either both islanders are lying or both are telling the truth.

Corollary 13: If you ask two islanders an objective question and they give different answers, then one is lying and one is telling the truth.
Proof: Suppose you ask two islanders an objective question and they give different answers. Suppose that they are both telling the truth. Since the answers are different then the answer's truth must be subject to the person who is answering the question. However, this is an objective question, so that cannot be. Also, suppose that they are both lying. Since they gave different answers, then the answer's truth must be subject to the person who is answering the question. However, this is an objective question, so that cannot be. Because they can neither both be lying nor both be telling the truth, then one must be lying and the other telling the truth.

Theorem 14: If you ask two islanders an objective question to which you do not know the answer, you will not be able to determine the truth simply by what the two islanders' answers are.
Proof: Suppose you ask two islanders an objective question to which you do not know the answer. If both islanders are truthtellers or both islanders are liars, they will both give you the same answer. You would not be able to determine the truth of the answer because you do not know the correct answer, so you do not know if both are lying or both are telling the truth. If the islanders are from different tribes, then one will say "Yes." and one will say "No.", but since you do not know the correct answer, you will not be able to determine which islander is lying and which is telling the truth. Therefore, you will not by the islander's answers be able to determine truth.

Corollary 15: To determine which tribe an islander is from only by asking her an objective question, the question asker must know the answer to the question.
Proof: Since you will not be able to determine which tribe an islander is from from an objective question to which you do not know the answer (Theorem 14) if you want to use an objective question it must be a question to which you know the answer.

Definition 16: To **know** the truth of something is to know if something is true or false because you already know the answer to a question. To **discover** something is to discern from the answer to a question or from another person's answer to the question that an answer is true or false.

Corollary 17: In order to discover the tribe of an islander, one must ask her a subjective question.
Proof: Since one cannot discover anything from an objective question to which one does not know the answer (**Theorem 14**) and if one did know the answer to the objective question, then they would know the tribe of the islander and would not be able to discover it. Therefore, one cannot discover anything with an objective question, and therefore in order to discover anything about the islanders, one must ask a subjective question.

Theorem 18: If you know the answer to a question that you ask an islander, then you know if the islander is tell the truth or lying.
Proof: Suppose you ask two islanders a question to which you already know the answer. If the islander says the answer you already know, then the islander is telling the truth. If the islander does not say the correct answer, then the islander is lying.

Corollary 19: If you know the tribe of an islander, then you know the answer to any question you ask her.
Proof: Suppose you ask an islander any question. Suppose you know which tribe the islander is from. If she is from the tribe of truthtellers, then the answer that that islander gives is the correct answer. Then that is the answer to the question. If instead you know that she is from the tribe of liars, then whatever that islander says is the wrong answer, and the opposite of that is the correct answer to the question.

Definition 20 For the remainder of the paper, the **situation** is that you as the question asker encounter two random islanders and ask them one question. An **outcome** is the resulting state after a question is asked and answered with regards to islanders' answers (Yes/No), the islanders' truthfulness (true/false), and your resulting knowledge of each islanders' tribal identity (knowing their tribe, discovering their tribe, or not knowing or discovering their tribe.) An outcome can be written in the form **YtNfkk** where the first letter represents the first islander's answer, the second letter represents the first islander's truthfulness, the third letter represents the second islander's answer, the forth letter represents the second islander's truthfulness, the fifth letter represents the resulting knowledge of the first islander, and the last letter represents the resulting knowledge of the second islander.

Theorem 21: There are at most 144 different outcomes of the situation.
Proof: Each of the islanders can each say either yes or no so there are two options each for spaces 1 and 3. Also, each islander could also be either lying or telling the truth so there are two options each for spaces 2 and 4.. Also, by the islanders' responses to the questions, you as the question asker can either know their tribe, discover their tribe, or not know or discover anything about their tribe so there are three options each for spaces 5 and 6. Therefore in regards to these three areas, there are at most 2x2x2x2x3x3=144 options for outcomes.

Theorem 22: There are at most 16 different outcomes that can arise from an objective question.

Proof: We know from Corollary 12 that if two islanders answer the question in the same way, then they are either both telling the truth or both lying. Therefore, if both islanders say "yes" there are two options, and if both islanders say "no" there are two options. Also, from Corollary 13, we know that if the islanders give different answers, then one is lying and one is telling the truth. Therefore, if one says "no" and the second one says "yes" then either the first one is lying and the second one is telling the truth or the first one is telling the truth and the second one is lying (2 options total). Alternatively, if the first one says "yes" and the second one says "no" then once again either the first one is lying and the second one is telling the truth or the first one is telling the truth and the second one is lying (2 options total). Thus, there are four possible combinations of "Yes" and "No" and each yields 2 possibilities of true or false. From Corollary 17, we know that nothing can be discovered from an objective question. Therefore, the only two options for the question asker's state of knowledge is that you either know which tribe each is from, or you do not know or discover which tribe each is from. If you know what tribe one is from, from Corollary 19 you know the answer to the question. And if you know the answer to the question, you know the tribe of the other islander (Theorem 18) Therefore, you either know the tribes of both islanders or you do not know the tribe of either islander. Therefore there are 4x2x2=16 different outcomes that could arise from asking an objective question.

These are:

- YtYtkk
- YtYtnn
- YfYfkk
- YfYfnn
- NtNtkk
- NtNtnn
- NfNfkk
- NfNfnn
- YtNfkk
- YtNfnn
- YfNtkk
- YfNtnn
- NfYtkk
- NfYtnn
- NtYfkk
- NtYfnn

Theorem 23: There are at most 64 different outcomes that reveal from which is tribe each islander.

Proof: To know which tribe each of the islanders is from, we simply want to either know or discover which tribe each islander is from. Each islander can still say "Yes" or "No" (2 options each for spaces 1 and 3) and they can be either lying or telling the truth (2 options each for spaces 2 and 4.) Then, for each islander, we want to either know or discover which tribe they are from (2 options each for spaces 5 and 6.) Therefore, there are 2x2x2x2x2x2=64 different outcomes in which the question asker could ascertain from which tribe was each islander.

Theorem 24: It is possible to know the answer to subjective questions.
Proof: Some examples are "Are you from the Island of Truthtellers and Liars?" "Do you have red hair?" "Are you male?"

Theorem 25: There are exactly 16 possibilities of combinations in which you would know the tribe of each islander
Proof: Each Islander can say either "Yes." or "No." Each islander could be telling the truth or lying.

Here is a list of outcomes in which the question asker would know the tribe of each islander and an example of a question for each one. For the examples, assume the first islander is a male with brown hair and the second islander is a female with red hair.

- YtYtkk "Is the sky blue?" "Yes." "Yes."
- YfYtkk "Do you have red hair?" "Yes." "Yes"
- NtYtkk "Do you have red hair?" "Yes." "Yes."
- NfYtkk "Is the sky blue?" "No." "Yes"
- YtYfkk "Are you a male?" "Yes." "Yes."
- YfYfkk "Is the sky purple?" "Yes" "Yes"
- NtYfkk "Is the sky purple?" "No." "Yes"
- NfYfkk "Are you a male?" "No." "Yes."
- YtNtkk "Are you a male?" "Yes." "No."
- YfNtkk "Is the sky purple?" "Yes." "No"
- NtNtkk "Is the sky purple?" "No." "No."
- NfNtkk "Are you a male?" "No." "No."
- YtNfkk "Is the sky blue?" "Yes." "No."
- YfNfkk "Do you have red hair?" "Yes." "Yes"
- NtNfkk "Do you have red hair?" "No." "No."
- NfNfkk "Is the sky blue?" "No." "No."

Theorem 26: There are four different types of questions that could together produce every outcome in which you would know the tribe of each islander. They are

1. Questions that you know are true for both islander
2. Questions that you know are false for both islander
3. Questions that you know are true for the first islander and false for the second islander
4. Questions that you know are false for the first islander and true for the second islander.

Proof: Question type one could produce YtYtkk, NfYtkk, YtNfkk, NfNfkk. This type is represented by the question "Is the sky blue?" above. Other questions could be, "Are you on the island of Truthtellers and Liars?" or "Am I asking you a question?" Question type two could produce YfYfkk, NtYfkk, YfNtkk, NtNtkk. This type is represented above by the question, "Is the sky purple?" Other questions could be, "Am I in Wisconsin?" or "Is your sister a boy?" Question type three could produce YtYfkk, NfYfkk, YtNtkk, NfNtkk. This type of question is represented above by the question "Are you male?" Another question could be "Are you wearing purple," assuming the first islander was wearing purple and the second islander wasn't. Question type four could produce YfYtkk, NtYtkk, YfNfkk, NtNfkk. This type is represented above by the question, "Do you have red hair?" Another question of this type could be, "Is your shoe untied?" Assuming that the second person's had a shoe that was untied and the first person did not. Since among these four questions all sixteen answer combinations can be reached depending on the tribes of the two islanders, then it is possible to produce every answer combination with only these four types of questions.

Theorem 27: Any question that satisfies the following conditions can be used to determine what tribe any two islanders are from:

1. A truthteller and a liar would always answer the question differently
2. You would be able to know or discover either what a truthteller would answer or what a liar would answer.

Proof: Suppose you have a question that a truthteller and a liar always answer differently. Suppose you knew or discovered how a truthteller would answer the question. Since the liar would answer it differently, then you would know that whatever the truthteller would say, the liar would say differently. Since you knew which answer a truthteller would say you would know that anyone who gave that answer was a truthteller and anyone who did not was a liar. Otherwise, if you instead know how a liar would answer the question, then when you asked the question, anyone who gave that answer was from the tribe of Liars and anyone who gave a different answer was from the tribe of Truthtellers. Since islanders must be from either the tribe of liars or the tribe of truthtellers, you would then be able to determine which tribe each islander was from.

Theorem 28: It is impossible to know something for one islander and discover something for the other islander with a single question

Proof: In order to discover something about either islander, we must ask a subjective question (Theorem 17.) Since you must know the answer to this question for one of the islanders, and since you have no prior knowledge of either islander, the question must involve something observable, such as hair color or gender. An answer that would be readily observable in one islander would most likely be readily observable in the other islander. Something that is readily observable in one islander and not in another will not become readily observable in the second as a result of that islander's answer to a question. Therefore, a question whose answer you know for one islander cannot be used to discover something for a different islander. Therefore, it is impossible to know something for one islander and discover something for the other islander with a single question.

Theorem 29: The following outcomes are impossible to reach with a single question:

YtYtkd	YfYtkd	NtYtkd	NfYtkd	YtYfkd	YfYfkd
NtYfkd	NfYfkd	YtNtkd	YfNtkd	NtNtkd	NfNtkd
YtNfkd	YfNfkd	NtNfkd	NfNfkd		

YtYtdk	YfYtdk	NtYtdk	NfYtdk	YtYfdk	YfYfdk
NtYfdk	NfYfdk	YtNtdk	YfNtdk	NtNtdk	NfNtdk
YtNfdk	YfNfdk	NtNfdk	NfNfdk		

Proof: These 16 outcomes all involve knowing something about one islander and discovering something about the other islander. From Theorem 28 we know this is impossible. Therefore, these are not possible outcomes.

Theorem 30: There are 16 outcomes in which the question asker would discover the tribe of each islander.
Proof: Each Islander can say either "Yes." or "No." Each islander could be telling the truth or lying.

Here is a list of outcomes and an example for each one. For the examples, assume the first islander is a male with brown hair and the second islander is a female with red hair.

- YtYtdd "Would a member of your tribe say that you are a truthteller?" "Yes." "Yes."
- YfYtdd "Would a member of your tribe say another member of your tribe would say that you have red hair?" "Yes" "Yes"
- NtYtdd "Would a member of your tribe say another member of your tribe would say that you have red hair?" "No." "Yes."
- NfYtdd "Would a member of your tribe say that you are a truthteller?" "No." "Yes."
- YtYfdd "Would a member of your tribe say another member of your tribe would say that you are a male?" "Yes." "Yes."
- YfYfdd "Would a member of your tribe say you are a liar?" "Yes." "Yes."
- NtYfdd "Would a member of your tribe say that you are a liar?" "No." "Yes."
- NfYfdd "Would a member of your tribe say another member of your tribe would say that you are a male?" "No." "Yes."
- YtNtdd "Would a member of your tribe say another member of your tribe would say that you are a male?" "Yes." "No."
- YfNtdd "Would a member of your tribe say that you are a liar?" "Yes." "No."
- NtNtdd "Would a member of your tribe say that you are a liar?" "No." "No."
- NfNtdd "Would a member of your tribe say another member of your tribe would say that you are a male?" "No." "No."
- YtNfdd "Would a member of your tribe say that you are a truthteller?" "Yes." "No."
- YfNfdd "Would a member of your tribe say another member of your tribe would say that you have red hair?" "Yes." "No."
- NtNfdd "Would a member of your tribe say another member of your tribe would say that you have red hair?" "No." "No."
- NfNfdd "Would a member of your tribe say you are a truthteller?" "No." "No."

Theorem 31: There are four different types of questions that could together produce every outcome in which you would discover the tribe of each islander. They are

1. Questions that you discover are true for both islander
2. Questions that you discover are false for both islander
3. Questions that you discover are true for the first islander and false for the second islander
4. Questions that you discover are false for the first islander and true for the second islander.

Proof:
Question type one could produce YtYtkk, NfYtkk, YtNfkk, NfNfkk. This type is represented by the question "Would a member of your tribe say that you are a truthteller?" above.
Question type two could produce YfYfkk, NtYfkk, YfNtkk, NtNtkk. This type is represented above by the question, "Would a member of your tribe say that you are a liar?"
Question type three could produce YtYfkk, NfYfkk, YtNtkk, NfNtkk. This type of question is represented above by the question "Would a member of your tribe say another member of your tribe would say that you are a male?"
Question type four could produce YfYtkk, NtYtkk, YfNfkk, NtNfkk. This type is represented above by the question, "Would a member of your tribe say another member of your tribe would say that you have red hair?"
Since among these four questions all sixteen answer combinations can be reached depending on the tribes of the two islanders, then it is possible to produce every answer combination with only these four types of questions.

Theorem 32: There are 32 possible outcomes in which the question asker would either know or discover the tribal identity of two islanders.
Proof: From Theorem 23 we know that there are at most 64 total outcomes in which the question asker would either know or discover the tribes of two islanders. We know from Theorem 29 that 32 of these are not possible. This leaves at most 32 left. We have presented these between Theorem 25 and Theorem 30. Therefore, these 32 all exist.