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Properties of Allocation Methods
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ABSTRACT

There are many different allocation methods for cooperative games. The fairness of these methods can be measured by the properties they possess. Because there is no "ideal method" that has all of these properties, deciding which method is the fairest is somewhat subjective. This project examines which methods have each property and includes some of the interesting proofs. It also discusses how some of these properties relate to one another.

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Introduction

An n-person cooperative game is a pair (N, v) where $N = \{1, 2, 3, \ldots n\}$ is the set of players and v is a real valued function on all coalitions $S \subseteq N$. Define the vector **x** = $(x_1, x_2, x_3, \ldots x_n)$ with real components to be an allocation of a game where x_j is the value being allocated to player j .

An *allocation method* is a function θ which, given any game (N, v), assigns an allocation **x** such that

$$\sum_N x_i = v(N) . \quad (\text{Note that } \theta_i(N,v) = x_i)$$

The latter condition is referred to as *efficiency*, and will be assumed for any allocation discussed in this paper.

There are many different allocation methods for cooperative games. The fairness of these methods can be measured by the properties they possess. Of course, because there is no "ideal method" that has all of the properties, deciding which method is the fairest is somewhat subjective. As seen in appendix A there is an argument for each property explaining why a method needs to have this particular property in order to be fair.

This paper will examine which methods have each property and will include some of the more interesting proofs. It will also discuss how some of these properties relate to one another.

SECTION I

	EAJV	EANV	PANV	SV	N	PCN
EFFICIENT	yes	yes	yes	yes	yes	yes
SYMMETRIC	yes	yes	yes	yes	yes	yes
PROPORTIONATE	yes	yes	yes	yes	yes	yes
COVARIANT	yes	yes	yes	yes	yes	yes
CONTINUOUS	yes	yes	no	yes	yes	yes
INDIV. RATIONAL	yes	no	yes	yes	yes	yes
GROUP RATIONAL	no	no	no	no	yes	yes
STABLE	no	no	no	no	yes	?
INDIV. REASONABLE	no	no	no	yes	yes	yes
GROUP REASONABLE	no	no	no	no	no	no
INDIV. SUB. FREE	no	no	yes	yes	yes	yes
S. CONSISTENT	no	no	no	no	yes	no
H & M CONSISTENT	no	no	no	yes	no	no
AGG. MONTONE	yes	yes	no	yes	no	yes
GROUP MONOTONE	yes	yes	no	yes	no	no
STRICT MONOTONE	no	no	no	yes	no	no
STRONG MONOTONE	no	no	no	yes	no	no
ADDITIVE	yes	yes	no	yes	no	no

The above chart specifies whether or not that specific method has the listed properties. The abbreviations are as follows:

EAJV - equal allocation of joint value, EANV - equal allocation of nonseparable value, PAJV - proportionate allocation of joint value, SV - Shapley value, N - nucleolus, and PCN - per capita nucleolus

Also note that any theorems or propositions that are referred to are contained within Appendix A.

Rationality

As we can see from Appendix A, rationality properties are important because if a method does not possess this property, a player or group may not be willing to cooperate. If the game is convex, we will see that it would be most beneficial to use the Shapley value since that would allow us to obtain a monotonic allocation as well as group rational. Otherwise, the nucleolus or per capita nucleolus would be sufficient enough to obtain group rationality along with maximum amount of other properties.

Equal allocation of joint value is individually rational since $\theta_i^{EAJV}(N,v) = v(i) + 1/n\,(v(N) - \sum_{j \in N} v(j))$, and by superadditivity we may conclude that $\sum_{j \in N} v(j) \leq v(N)$. Thus, $\theta_i^{EAJV} \geq v(i)$.

Equal allocation of nonseparable value is not individually rational because if we consider the game v(123) = v(23) = 1, v(12) = v(13) = v(1) = v(2) = v(3) = 0, then $\theta^{EANV}(N,v) = (-1/3, 2/3, 2/3)$. Notice that $-1/3 = \theta_i^{EAJV} \leq v(1) = 0$. This example can also be used to show that equal allocation of nonseparable value is not group rational, since there exists a nonempty core. For example the point (0, 1/2, 1/2) is in the core of this game.

Proportionate allocation of joint value is individually rational due to the fact that, $\theta_i^{PAJV}(N,v) = v(i) + \frac{r_i}{\sum_{j \in N} r_j}\,[\, v(N) - \sum_{j \in N} v(j)] \geq v(i)$, because $\frac{r_i}{\sum_{j \in N} r_j} \geq 0$ and $\sum_{j \in N} v(j) \leq v(N)$ due to superadditivity.

By their definition, we can state that the nucleolus and the per capita nucleolus are individually rational.

Because $\sum_{S \subseteq N} \frac{(s-1)!(n-s)!}{n!} = 1$ and $V(S) - v(S-i) \geq v(i)$ for all $S \subseteq N$, the Shapley value is individually rational.

Equal allocation of joint value does not possess group rationality. This can be proven by considering the veto power game v(1) = v(2) = v(3) = v(12) = 0, v(13) = v(23) = v(123) =1. The only group rational point for this game is **x** = (0, 0, 1), however, $\theta_i^{EAJV}(N,v) = (1/3, 1/3, 1/3)$. Therefore, we can conclude that this method is not group rational.

Considering the four player game v(1) = v(2) = v(3) = v(4) = v(14) = v(23) = 0, v(12) = v(13) = v(24) = v(34) = v(124) = v(134) = v(234) = 1, and v(123) = v(N) = 2.

$\theta^{PAJV}(N,v) = (2/3, 2/3, 2/3, 0)$, but the only group rational point is (0, 1, 1, 0) therefore proportionate allocation of joint value is not group rational.

The Shapley value makes the allocation $\phi = (.167, .167, .667)$ for the players in the veto power game. The only group rational point in this game is (0, 0, 1), therefore the Shapley value is not group rational. We can also note, by theorem 3, that the Shapley value is not group rational because it is group monotone.

If the core(N,v) is not empty, there exists an imputation **x** with a nonpositive excess vector. Because the nucleolus minimizes the maximum excess and the per capita nucleolus uses the same method with the excess divided by a positive value, they must also have a nonpositive excess vector. This would imply that the nucleolus and the per capita nucleolus is in the core of the game. Therefore, they are both group rational.

Stability

Without stability a method may not be able to yield an allocation that is free from credible objections. This could cause a problem between two or more players because if one player is not convinced that the chosen allocation is the best for him, he may not be willing to cooperate. Of course, if an allocation is in the core of the game then this would be stable.

Equal allocation of joint value is not a stable method. This can be proven using the veto power game, $v(1) = v(2) = v(3) = v(12) = 0$ and $v(13) = v(23) = v(123) = 1$. $\theta^{EAJV}(N,v) = (1/3, 1/3, 1/3)$, but player three can make a complaint to either player 1 or 2 without these two players being able to counterobject. For example, player three could choose the allocation $\mathbf{y} = (1/3, 0, 2/3)$ with the coalition $\{1, 3\}$, and player 2 could not counterobject since to do so he would have to cooperate with player 3.

Using the game $v(1) = v(2) = v(3) = v(12) = v(13) = 0$ and $v(23) = v(123) = 1$, $\theta^{EANV}(N,v) = (-1/3, 2/3, 2/3)$. So, equal allocation of nonseparable value is not stable because in this game player 1 could complain against player 2 using the coalition, $S = \{1\}$

and the allocation $\mathbf{y} = (0, 1/3, 2/3)$. Player 2 can not object since $v(2) \leq y_2$, $v(12) \leq y_1 + y_2$, and $v(23) \leq y_2 + y_3$. Also, player 3 could not object because $v(3) \leq y_3$, $v(13) \geq y_1 + y_3$, and $v(23) \leq y_2 + y_3$.

Examining the game used to show that proportionate allocation of joint value is not group rational, we can see that this game also shows that this method is not stable. For example, $\theta^{PAJV}(N,v) = (2/3, 2/3, 2/3, 0)$ which means player 2 could make a complaint against player 1 with the allocation $\mathbf{y} = (0, 1, 1, 0)$ and the coalition $S = \{2, 3\}$. Because the allocation $\mathbf{y} \in core(N,v)$, we can conclude that no counterobjection can be made.

Using the veto power game once again, we can show that the Shapley value is not stable. Because $\phi = (1/6, 1/6, 1/3)$, player 3 could make a complaint against either players 1 or 2 using the allocation $\mathbf{y} = (0, 1/6, 5/6)$ with the coalition $S = \{2, 3\}$ or the allocation $\mathbf{y} = (1/6, 0, 5/6)$ with the coalition $S = \{1, 3\}$, respectively. Neither player would be able to counterobject without using player three in the process.

The following proof will exhibit that the nucleolus is a stable method. Suppose there is a game (N,v) that the nucleolus assigns the allocation $\nu = (x_1, x_2, \ldots x_n)$ where $x_1, x_2, \ldots x_n$ are real values satisfying efficiency. Suppose $\mathbf{x}$ is not stable. Then there exists an objection $(\mathbf{y}, S)$ of a player i against a player j for which there is no counterobjection, $(\mathbf{z}, T)$. If there is a coalition T satisfying $i \notin T$, $j \in T$, and $e(\mathbf{x}, S) \leq e(\mathbf{x}, T)$, then the allocation $\mathbf{z}$ is defined by

$$\mathbf{z} = \begin{cases} y_k, & \text{if } k \in S \cap T \\ x_k, & \text{if } k \in T \backslash S \\ \delta, & \text{if } k \notin T \end{cases}$$

where δ is defined so that $\sum_{k \in N} z_k = v(N)$, forms a counterobjection with T. Since it was assumed that no such counterobjection exists, $e(\mathbf{x}, S) > e(\mathbf{x}, T)$ for all T satisfying $i \notin T$ and $j \in T$. So, increasing x_i and decreasing x_j by small amounts would cause $e(\mathbf{x})$ to decrease lexicographically. Therefore, $\mathbf{x}$ cannot be the nucleolus, which is a contradiction. Thus, the nucleolus is a stable method.

Reasonableness

Equal allocation of joint value is not individually reasonable since $\theta^{EAJV}(N,v) = (5/3, 5/3, 5/3)$ for the game $v(i) = 1$, $v(12) = v(13) = 2$, $v(23) = 4$, and $v(N) = 5$. This would mean that the $\max(v(S) - v(S-1))$, for all $S \ni 1$, is $v(N) - v(23) = 1 \leq 5/3 = x_1$. This game also shows that equal allocation of joint value is not group reasonable because although there exists a nonempty set of group reasonable point $(5/3, 5/3, 5/3)$ is not in this set. For example, $(1, 2, 2)$ is in the set of group reasonable points.

The game $v(i) = 0$ for all $i \in N$, $v(ij) = 5$ for all $i \neq j \in N$, $v(123) = v(124) = 9$, $v(134) = 8$, $v(234) = 5$, and $v(N) = 10$ proves that equal allocation of nonseparable value is not individually reasonable. This is because $\theta^{EANV}(N,v) = (21/4, 9/4, 5/4, 5/4)$ which makes the $\max(v(S) - v(S-i))$, for all $S \ni i$, $v(N) - v(234) = 5 \leq 21/4 = x_1$. As stated above, because a set of group reasonable points exist for this game, for example $(7/2, 5/2, 2, 2)$ is group reasonable, equal allocation of nonseparable value is not contained in this set.

Proportionate allocation of joint value allocates $\theta_1^{PAJV}(N,v) = 25/4$ for the game $v(i) = 0$ for all $i \in N$, $v(ij) = 5$ for all $i \neq j \in N$, $v(123) = v(124) = 9$, $v(134) = 8$, $v(234) = 5$, and $v(N) = 10$. Therefore, because $x_1 = 25/4 \geq 5 = v(N) - v(234)$ this method is not individually nor group reasonable.

The Shapley value is individually reasonable because it can never be greater than the maximum marginal contribution. This is due to the fact that the Shapley value determines all of a players marginal contributions and then takes the weighted average of these values.

By the following proof, the nucleolus is individually reasonable:

Proof:

Suppose that the nucleolus is not individually reasonable, then there exists an $i \in N$ such that,

$$x_i > \max\{v(S) - v(S\text{-}i), \text{ for all } S \ni i\} \Rightarrow x_i > v(S) - v(S\text{-}i), \text{ for all } S \ni i,$$

but $v(S) - v(S\text{-}i) = e(\mathbf{x}, S) - e(\mathbf{x}, S\text{-}i) + x_i \Rightarrow e(\mathbf{x}, S\text{-}i) > e(\mathbf{x}, S)$.

Then consider the allocation **y** such that $y_j = x_j + \varepsilon$, for all $j \neq i \in N$, and $y_i = x_i - (n-1)\varepsilon$, where ε is small enough to preserve the order in the excess vector, $e(\mathbf{x})$. This implies that $e(\mathbf{y}, S\text{-}i) < e(\mathbf{x}, S\text{-}i)$, therefore **y** is lexicographically less that **x**, therefore **x** could not have been the nucleolus which is a contradiction. So, we conclude that the nucleolus is individually reasonable.

Using the same reasoning we can conclude that the per capita nucleolus is also individually reasonable since if the per capita nucleolus is not individually reasonable, then there exists an $i \in N$ such that $e(\mathbf{x}, S\text{-}i) > e(\mathbf{x}, S)$ for all $S \subseteq N$. If the largest excess (which cannot contain i) is nonnegative, then for the corresponding coalition {S-i},

$$\frac{e(\mathbf{x}, S\text{-}i)}{|S\text{-}i|} > \frac{e(\mathbf{x}, S)}{|S|}$$

and so a **y** can be constructed as in the previous paragraph for which $e(\mathbf{y}) < e(\mathbf{x})$. This contradiction to **x** being the per capita nucleolus implies that all of the excesses are negative, that is, **x** is group rational. But the $x_i = v(N) - \sum_{j \neq i} x_i \leq v(N) - v(N\text{-}i)$, and so **x** is individually reasonable.

Because of the game $v(N) = 10$, $v(S) = 5$ if $S = \{4,5\}$ or $|S| \geq 3$, and $v(S) = 0$ otherwise, the Shapley value is not group reasonable. (1.833, 1.833, 1.833, 2.25, 2.25) is the allocation given by the Shapley value which implies that $x_1 + x_2 + x_3 = 5.5 \geq 5 = \max \{v(S) - v(S\text{-}i)\text{: for all } S \supseteq \{1, 2, 3\}\}$. Of course, there does exist a group rational point for this game given by the nucleolus, $\nu = (5/3, 5/3, 5/3, 5/2, 5/2)$.

Considering the 6-player game:

$$v(1) = v(4) = v(6) = 0.2 \quad v(2) = v(3) = v(5) = 0$$

$$v(12) = v(13) = v(24) = v(34) = v(15) = v(45) = v(26) = v(36) = v(56) = 0.2$$

$$v(23) = v(25) = v(35) = 0 \qquad v(14) = v(16) = v(25) = v(46) = 0.4$$

$$v(123) = 0.2 \quad v(124) = 0.4 \quad v(125) = 3.1 \quad v(126) = 0.4 \quad v(134) = 3.1 \quad v(135) = 0.2$$

$$v(136) = 0.4 \quad v(145) = 0{,}4 \quad v(146) = 3.1 \quad v(156) = 0.4 \quad v(234) = 0.2 \quad v(235) = 0.0$$

$v(236) = 3.1$ $v(245) = 3.1$ $v(246) = 3.1$ $v(256) = 0.2$ $v(345) = 0.2$ $v(346) = 0.4$

$v(356) = 3.1$ $v(456) = 0.4$ $v(1234) = 3.1$ $v(1235) = 3.1$ $v(1236) = 3.3$

$v(1245) = 3.3$ $v(1246) = 3.1$ $v(1256) = 3.3$ $v(1345) = 3.1$ $v(1346) = 3.3$

$v(1356) = 3.3$ $v(1456) = 3.1$ $v(2345) = 3.1$ $v(2346) = 3.3$ $v(2356) = 3.1$ $v(2456) = 3.3$

$v(3456) = 3.3$ $v(12345) = 2.75$ $v(12345) = 3.3$ $v(12346) = 3.5$ $v(12356) = 3.3$

$v(12456) = 3.5$ $v(13456) = 3.5$ $v(23456) = 3..2$ $v(123456) = 6$

the nucleolus and per capita nucleolus yield the allocation $\nu = \nu^{PC} = (1, 1, 1, 1, 1, 1)$. However, this is not a group reasonable allocation. This is because the max $\{v(S) - v(S-\{235\})\} = 2.9$, however $x_2 + x_3 + x_5 = 3$. Note that the Shapley value $\phi = (1.067, 0.927, 0.920, 1.067, 0.927, 1.093)$ is a group reasonable point. We conclude that the nucleolus and the per capita nucleolus are not group reasonable.

Consistency

By theorem 5, the Shapley value is the only method that is Hart & Mas-colell consistent. Also, because of theorem 4, the nucleolus is the only method which is Sobolev consistent. This means all of the other methods discussed in this paper are not Hart & Mas-colell consistent.

Monotonicity

Many times in real life situations the values assigned to a given player may change, therefore monotonicity may play a big factor in any decision. Of course, any player who receives less that they previously did, while their value has increased will not consider the allocation fair. In this type of situations we will see that the Shapley value would be the best method to make the allocation because it is easy to show that the Shapley value satisfies all of the monotonicity properties, and by theorem 2, none of the other methods are strongly monotone.

Equal allocation of joint value is aggregate monotone since, $\theta_i^{EAJV}(N,v) = v(i) + \frac{1}{n}(v(N) - \sum_{j \in N} v(j))$ can not decrease if $v(N)$ or any $v(S)$, where $S \ni i$, is increased. This also implies that equal allocation of joint value is group monotone. Also, if we substitute $v(N) - v(N-i)$ for s_i in the formula for equal allocation of nonseparable value, then $\theta_i^{EANV}(N,v)$ can not decrease when $v(N)$ or any $v(S)$, where $S \ni i$, is increased. In other words, $\theta_i^{EANV}(N,v) = v(N) - v(N-i) + \frac{1}{n}v(N) - v(N) - \frac{1}{n}\sum_{i \neq j \in N} v(S-j))$ which reduces to $\frac{1}{n}v(N) + \frac{1-n}{n}v(N-i) + \frac{1}{n}\sum_{i \neq j \in N} v(N-j)$. In the last equation we can see that the only terms with a negative value are the terms that do not contain i. Therefore, equal allocation of nonseparable value is aggregate and group monotone. Now, using the following counterexample it is easy to see that equal allocation of joint value is not strictly monotone. If we consider a game similar to the veto power game only $v(1)$ has increased to one, the allocation given by this method is still, $\theta^{EAJV}(N,v) = (1/3, 1/3, 1/3)$. By the same reasoning, equal allocation of nonseparable value is also not strictly monotone since on the revised game just described $\theta^{EANV}(N,v) = (1/3, 1/3, 1/3)$.

By the following proof, the per capita nucleolus is aggregate monotone:

Proof: Consider two games (N, u) and (N, v) such that $v(N) = u(N) + \varepsilon$, and $v(S) = u(S)$ otherwise. If the per capita nucleolus allocates $\mathbf{x} = \nu^{PC}(N, u) = (x_1, x_2, \ldots x_n)$ then $\mathbf{y} = \nu^{PC}(N, v) = (x_1+\varepsilon/N, x_2+\varepsilon/N, \ldots x_n+\varepsilon/N)$. Since pce(x) is lexicographically minimal, then pce(y, S) must be lexicographically minimal because pce(y, S) = pce(x,S) + ε/N, and ε/N is just a constant. However, by theorem 3, the per capita nucleolus is not group monotone because it is group rational. This also implies that this method is not strictly monotone. This is easily displayed by the following example, recall the game $v(1) = v(2) = v(3) = v(4) = v(23) = v(14) = v(23) = 0$, $v(12) = v(13) = v(24) = v(34) = v(134) = v(234) = v(124) = v(123) = 1$, and $v(1234) = 2$. The per capita nucleolus of this game is $\nu^{PC} = (1/2, 1/2, 1/2, 1/2)$, however if we increase $v(123)$ to 2 then $\nu^{PC} = (0, 1, 1, 0)$. We can see that although $v(123)$ increased, player 1's allocation decreased. Not only does this example exhibit that the per capita nucleolus is not strict monotone, but also that it is not group monotone.

Proportionate allocation of joint value is not aggregate monotone and this is shown by the veto power game $v(1) = v(2) = v(3) = 0$, $v(12) = 6$, $v(13) = v(23) = 24$, $v(123) = 30$. $\theta^{PAJV}(N,v) = (5, 5, 20)$. Now, if we look at the same game (N,v) except v(N) is increased to 31, then $\theta^{PAJV}(N,v) = (5.564, 5.564, 19.872)$. Notice that although v(N) has increased x_3 has decreased since $19.872 < 20$. By proposition 2, the above statement implies that this method is also not group or strict monotone.

Using the following two six player games we can see that the nucleolus is not aggregate monotone. Consider the game:

$v(i) = 0$, for all $i \in N$ $v(ij) = 0$, for all $i \neq j \in N$

v(123) = 0.75 v(124) = 0.50 v(125) = 0.65 v(126) = 0.35 v(134) = 0.25

v(135) = 0.25v(136) = 0.85 v(145) = 0.55 v(146) = 0.25 v(156) = 0.65

v(234) = 0.45 v(235) = 0.15v(236) = 0.45 v(245) = 0.85 v(246) = 0.50

v(256) = 0.25 v(345) = 0.654 v(346) = 0.0 v(356) = 0.24 v(456) = 0.63

v(1234) = 1.50 v(1235) = 1.85 v(1236) = 1.65 v(1245) = 1.654 v(1246) = 1.50

v(1256) = 1.65 v(1345) = 1.35 v(1346) = 1.45 v(1356) = 1.6 v(1456) = 1.66

v(2345) = 1.68 v(2346) = 1.55 v(2356) = 1.85 v(2456) = 1.33 v(3456) = 1.77

v(12345) = 2.75 v(12345) = 2.26 v(12346) = 2.56 v(12356) = 2.95

v(12456) = 2.65 v(13456) = 2.75 v(23456) = 1.33 v(123456) = 4

The nucleolus for this game is $\nu(N,v) = (0.625, 0.625, 0.695, 0.525, 0.785, 0.745)$, and if we consider the same game and increase v(N) by 0.1 the nucleolus yields $\nu = (0.675, 0.675, 0.725, 0.575, 0.725, 0.725)$. Notice that the amount allocated to players 5 and 6 have decreased even though there was an increase in v(N). Thus, we can conclude that the nucleolus is not aggregate monotone, and hence it is not group monotone nor strictly monotone.

Additive

It is shown directly that equal allocation of joint and nonseparable value are additive. Consider the following: two games (N,v^1) and (N,v^2) then,

$$\theta_i^{EAJV}(N,v^1+v^2) = (v^1+v^2)(i) + \frac{1}{n}((v^1+v^2)(N) - \sum_{i \neq j \in N} (v^1+v^2)(j))$$

$$= v^1(i) + v^2(i) + \frac{1}{n}(v^1(N) + v^2(N) - \sum_{i \neq j \in N} (v^1(j) + v^2(j)))$$

$$= \theta_i^{EAJV}(N,v^1) + \theta_i^{EAJV}(N,v^2)$$

$$\theta_i^{EANV}(N,v^1+v^2) = (v^1+v^2)v(N) - (v^1+v^2)v(N-i) + \frac{1}{n}((v^1+ v^2)(N) - \sum_{i \neq j \in N} (v^1+v^2(N) - v^1+v^2(N-i))$$

$$= v^1(N) + v^2(N + v^1(N-i) + v^2(N-i) + \frac{1}{n}(v^1(N)+ v^2(N) - \sum_{i \neq j \in N} (v^1(N)+ v^2(N) - v^1(N-i)-v^2(N-i))$$

$$= s_i^1 + s_i^2 + \frac{1}{n}(v^1(N)+ v^2(N) - \sum_{i \neq j \in N} s_j^1+s_j^2)$$

$$= \theta_i^{EANV}(N,v^1) + \theta_i^{EANV}(N,v^2)$$

By the following example proportionate allocation of joint value is not additive:

$v^1(i) = 0$ for all $i \in N$, $v^1(12) = 1/4$, $v^1(13) = 3/4$, $v^1(23) = 1/4$, $v^1(N) = 1$

$v^2(i) = 1$ for all $i \in N$, $v^2(ij) = 1$ for all $i \neq j \in N$, $v^2(N) = 3$

$v^3(i) = 1$ for all $i \in N$, $v^3(12) = 5/4$, $v^3(13) = 7/4$ $v^3(23) = 5/4$

$v^3(N) = 4$, where $v^3 = v^1 + v^2$

$\theta^{PAJV}(N,v^1) = (3/7,1/7,3/7)$ $\theta^{PAJV}(N,v^2) = (1,1,1)$ $\theta^{PAJV}(N,v^3) = (11/4,9/4,11/4)$

Notice that $\theta(v^1) + \theta(v^2) \neq \theta(v^3)$, therefore PAJV is not additive.

The Shapley value is additive because given three different games v^1, v^2, and $v^3 = v^1+v^2$,

$$\phi^1 = \sum_{S \subseteq N} \frac{(s-1)!(n-s)!}{n!}(v^1(S) - v^1(S-i)),\ \phi^2 = \sum_{S \subseteq N} \frac{(s-1)!(n-s)!}{n!}(v^2(S) - v^2(S-i)),$$ and

$$\phi^{1+2} = \sum_{s \subseteq N} \frac{(s-1)!(n-s)!}{n!}((v^1(S) - v^1(S-i) + (v^2(S) - v^2(S-i)) = \phi^1 + \phi^2$$

Considering the same three games used to show that proportionate allocation of joint value is not additive, we can show that these examples also prove that the nucleolus and per capita nucleolus are also not additive. In the first game the nucleolus yields $\nu(N, v^1) = (.438, .125, .438)$ while the per capita nucleolus yields $\nu^{PC}(N, v^1) = (.458, .083, .458)$, and both methods give the allocation $\nu(N, v^2) = \nu^{PC}(N, v^2) = (1, 1, 1)$ for the second. However, $\nu(N,v^3) = \nu^{PC}(N,v^3) = (1.333, 1.333, 1.333)$ which is not equal to the sum of the allocations for the first two games for either method.

SECTION II

The following result establishes a relationship between these group rationality and group monotonicity.

Theorem: (Young, 1985) There exists no allocation method that is group rational and group monotone on games of $|N| \geq 5$.

The following results complement and generalize Young's Theorem.

Theorem: There exists no allocation method that is group rational and group monotone on four player games.

Proof: Consider a game (N,v) where $N = \{1, 2, 3, 4\}$ and

$v(1) = 0$	$v(12) = 1$	$v(123) = 1$
$v(2) = 0$	$v(13) = 1$	$v(124) = 1$
$v(3) = 0$	$v(14) = 0$	$v(134) = 1$
$v(4) = 0$	$v(23) = 0$	$v(234) = 1$
	$v(24) = 1$	$v(N) = 1$
	$v(34) = 1$	

It is easily seen that the core(N,v) $\neq \emptyset$ and also consists of more that one point. Now consider four different games, (N,v^1), (N,v^2), (N,v^3), and (N,v^4) which are the same as our original game except v(123), v(234), v(124), and v(134) are increased by one, respectively. In each new game the core has a unique point:

$$\text{core}(N, v^1) = \{(0, 1, 1, 0)\}$$
$$\text{core}(N, v^2) = \{(0, 1, 1, 0)\}$$
$$\text{core}(N, v^3) = \{(1, 0, 0, 1)\}$$
$$\text{core}(N, v^4) = \{(1, 0, 0, 1)\}$$

For instance, suppose $\mathbf{x} \in \text{core}(N, v^1)$. If $x_4 > 0$ then $x_1 + x_2 + x_3 < 2 = v(123)$; so, $x_4 = 0$. Now, if $x_2 < 1$ then $x_2 + x_4 < v(24)$, and if $x_3 < 1$ then $x_3 + x_4 < v(34)$; so, $x_2 \geq 1$ and $x_3 \geq 1$. Therefore, we conclude that $\text{core}(N, v^1) = \{(0, 1, 1, 0)\}$. A similar argument can be given for each of the remaining games. Thus any allocation method satisfying group rationality would choose these unique points.

Notice that in game (N, v^1) although v(123) has increased $\theta_1(N, v^1) = 0$. This implies that in order for our method to also satisfy group monotonicity the amount being allocated to player 1 must be zero. Similarly, game (N, v^2) implies the amount allocated to player 4 in the original game (N, v) is zero, game (N, v^3) implies the amount that must be allocated to player 2 in the game (N, v) is zero, and game (N, v^4) implies the amount that must be allocated to player 3 in the game (N, v) is zero.

Thus, in order for our method to satisfy group rationality and group monotonicity the allocation of our original game (N, v) must be $\mathbf{x} = (0, 0, 0, 0)$. However, this violates our efficiency assumption. Therefore, no allocation method can satisfy both group rationality and group monotonicity on our game (N, v).

Theorem: There is an infinite number of allocation methods which are group rational and group monotone on three player games.

Before a formal proof may be given, the following definitions and theorems need to be stated.

The preimputation set of a game (N, v) is any allocation $\mathbf{x} = (x_1, x_2, x_3, \ldots x_n)$ such that efficiency is satisfied, $\sum_N x_i = v(N)$.

Define the excess of coalition S to be

$$e(x, S) = \frac{\left[v(S) - \sum_{i \in S} x_i\right]}{|w|_S}$$

where $\mathbf{w}$ is a given vector consisting of positive values dependent on the size of the coalition, S. Then, let $e(\mathbf{x})$ be the vector of excesses ordered from largest to smallest. The w-prenucleolus is the preimputation that lexicographically minimizes $e(\mathbf{x})$ over the set of preimputations. The w-nucleolus is the imputation that lexicographically minimizes $e(\mathbf{x})$ over the set of imputations. For example, the nucleolus (Schmiedler) has a $\mathbf{w} = (1, 1, 1)$ on three player games, and the per capita nucleolus (Grotte) has a $\mathbf{w} = (1, 2, 3)$ on three person games.

Let $\beta = \{S_1, S_2, S_3, \ldots S_m\}$ be a collection of subsets of $N = \{1, 2, 3, \ldots n\}$. β is N-balanced if we can find a balancing vector $\mathbf{y} = (y_1, y_2, \ldots y_m)$ such that, for every player i,

$$\sum_{j:\ i \in S_j} y_j = 1, \text{ and all } y_j > 0.$$

Given an imputation **x**, let β_k be the set of all coalitions with k^{th} maximal excess, such that the excess of β_i is greater than the excess of β_{i+1}. Define the array determined by **x**,

$$C_k = \bigcup_{i=1}^{k} \beta_i$$

For each set of coalitions $\{S_1, S_2 \ldots S_m\}$ with k^{th} maximal excess in β_k the following equalities hold: $e(\mathbf{x}, S_i) = e(\mathbf{x}, S_j)$ where $S_i, S_j \in \beta_k$.

The following is a slight generalization of Kohlberg's Theorem.

Theorem: An allocation **x** is the w-prenucleolus if the the array of collections $C_1, C_2, \ldots C_{k^*}$ determined by **x** consists of only balanced collections.

If the vector **x** is the w-prenucleolus, then all of the collection C_k, made up of the sets of excesses, β_i, will be balanced. In order to determine whether or not **x** is the w-prenucleolus, we only need to check the collections up to the first k^* sets of excesses, where k^* is the point where **x** can be uniquely determined. Suppose we have a three player game with excess vector $e(\mathbf{x})$. Then we can define the following five cases for β_1 and β_2:

	β_1	β_2
1.	{ 1, 2, 3}	
2.	{ 12, 13, 23}	
3.	{ 1, 23}	{ 2, 3}
4.	{ 1, 23}	{ 12, 13}
5.	{ 1, 23}	{ 2, 13}

Of course, other permutations exist, but only bring about something symmetric to the above cases. Therefore, we can say that these five cases make up all possible cases because of the following:

Since $C_1 = \beta_1$, β_1 must be balanced. This implies that β_1 contains a minimally balanced set {1, 2, 3}, {12, 13, 23}, or {1, 23}.

(1) If β_1 contains {1, 2, 3} the **x** is defined as follows:

$$x_1 = \frac{v(N) + 2v(1) - v(2) - v(3)}{3}, \quad x_2 = \frac{v(N) + 2v(2) - v(1) - v(3)}{3},$$

$$x_3 = \frac{v(N) + 2v(3) - v(1) - v(2)}{3}$$

(2) If β_1 contains {12, 13, 23} then **x** is defined as follows:

$$x_1 = \frac{v(N) + v(12) + v(13) - 2v(23)}{3}, \quad x_2 = \frac{v(N) + v(12) + v(23) - 2v(13)}{3},$$

$$x_3 = \frac{v(N) + v(13) + v(23) - 2v(12)}{3}$$

(3) If β_1 contains {1, 23} then one of the following is true:

a. it also contains {2, 3} OR β_2 contains {2, 3} and **x** is defined by:

$$x_1 = \frac{\left(\frac{w_1}{w_2} + 1\right)v(N) + 2v(1) + 2v(2) - v(3) - \frac{2w_1}{w_2}v(23)}{\frac{w_1}{w_2} + 3},$$

$$x_2 = \frac{v(N) + \left(\frac{w_1}{w_2} + 1\right)(v(2) - v(3)) + \frac{w_1}{w_2}v(23)}{\frac{w_1}{w_2} + 3},$$

$$x_3 = \frac{v(N) + \frac{w_1}{w_2}v(23) + v(23) + 2v(3) - v(1) - 2v(2)}{\frac{w_1}{w_2} + 3}$$

b. it also contains {12, 13} OR β_2 contains {12, 13} and **x** is defined by:

$$x_1 = \frac{w_1 v(N) + w_2 v(1) - w_1 v(23)}{w_1 + w_2},$$

$$x_2 = \frac{w_2 v(N) - v(1) + v(23) + (w_1 + w_2)(v(12) - v(13))}{2(w_1 + w_2)},$$

$$x_3 = \frac{w_2 v(N) - v(1) + v(23) + (w_1 + w_2)(v(13) - v(12))}{2(w_1 + w_2)}$$

c. it also contains {2, 13} OR β_2 contains {2, 13} and **x** is defined by:

$$x_1 = \frac{w_1 v(N) + w_2 v(1) - w_1 v(23)}{w_1 + w_2}, \quad x_2 = \frac{w_1 v(N) + w_2 v(2) - w_1 v(13)}{w_1 + w_2},$$

$$x_3 = \frac{(w_2 - w_1)v(N) - w_2(v(1) + v(2)) + w_1(v(13) + v(23))}{w_1 + w_2}$$

In all of these cases **x** is uniquely determined. If we assume that $w_2 \geq w_1$ then for each of the previous equations for x_i do not have any negative coefficients of coalitions S, which contain that specific player, i. Therefore, we may conclude that if coalitions containing player i increase, then x_i cannot decrease. Thus, since the **w**-prenucleolus must be one of these five cases of the **w**-prenucleolus we can conclude that the w–prenucleolus is monotone on three player games.

Next, we show not only that the **w**-prenucleolus is group rational. Consider a game (N, v) such that core(N,v) is not empty and allocation **x** , which is in the core of the game. Because **x** is in the core of (N, v), by definition, we can say that $e(\mathbf{x}, S) \leq 0$ for all $S \subseteq N$, therefore $e(\mathbf{x})$ consists of all nonpositive values. However, since, ν, the w-prenucleolus lexicographically minimizes $e(\nu)$, then $e(\nu) \leq e(\mathbf{x})$ which implies that $e(\nu)$ consists of nonpositive values. Thus, we can conclude that the **w**-prenucleolus is in the core of (N, v), and therefore is group rational.

In order to prove that the **w**-prenucleolus is an infinite class of allocation methods, consider the three player game, (N, v) where all singletons have a value of zero, v(i) = 0 for all $i \in N$, and v(12) = v(13) = 1, v(23) = 2, and v(N) = 4. If we define $\mathbf{w} = (1, w_2, w_3)$ where $1 \leq w_2 \leq 2$, then the **w**-prenucleolus of the game (N, v) is:

$$\nu_w(v,N) = \left(\frac{2}{1+w_2}, \frac{2+w_2}{1+w_2}, \frac{2+w_2}{1+w_2}\right).$$

These values were obtained from solving the following two consecutive linear programs:

min α

$$\begin{aligned}
\text{s. t.}\quad -x_1 \qquad\qquad &\leq \alpha \\
-x_2 \qquad &\leq \alpha \\
-x_3 &\leq \alpha \\
1 - x_1 - x_2 \qquad &\leq w_2\alpha \\
1 - x_1 \qquad - x_3 &\leq w_2\alpha \\
2 \qquad - x_2 - x_3 &\leq w_2\alpha
\end{aligned}$$

By adding the first and last inequalities and subtracting the equality constraint his LP gives a lower bound for $\alpha \geq \frac{-2}{1+w_2}$, therefore, we can see by substitution of our solution that we are at optimal solution, directly yielding our value for x_1.

$x_1 + x_2 + x_3 = 4$

$$
\begin{array}{llll}
\min \ \beta & & & \\
\text{s. t.} & -x_1 & = & \alpha \\
& -x_2 & \leq & \beta \\
& -x_3 & \leq & \beta \\
& 1 - x_1 - x_2 & \leq & w_2\beta \\
& 1 - x_1 \quad - x_3 & \leq & w_2\beta \\
& 2 \quad - x_2 - x_3 & = & w_2\alpha
\end{array}
$$

By substituting our value for x_1, it is easily seen that this LP directly yields our values for x_2 and x_3.

$x_1 + x_2 + x_3 = 4$

Because our value of w_2 may be any value on the interval $[1, 2]$, which would yield different solutions for the w-prenucleolus, we conclude that the w-prenucleolus is an infinite class of games.

This leads us to our overall conclusion, there exists an infinite class of games that are group rational and group monotone on three player games, namely, the w–prenucleolus.

It is straight-forward to extend these arguments to show that the w-nucleolus is also an infinite class of methods that are group rational and monotone on three player games. Indeed, the last two arguments (group rational and infinite class) are identical. Only the group monotonicity must be checked by considering a number of new cases corresponding to one or more the the x_i's being equal to $v(i)$.

Further Study Suggestions

In this paper, we discussed the different properties that each method possesses. We also talked about how group rationality and monotonicity relate to each other. Are their other properties that directly or indirectly relate to one another in some way? Questions for possible explorations are:

– Can games that are aggregate monotone be classified in any way?

– Can games for which the nucleolus is group monotone be classified? (Possibly by looking at the β's, sets of coalitions determined by the excesses, which contain subsets of another coalition.)

– Can stability and group rationality somehow be related?

– Can the Shapley value be classified uniquely by somehow using strict monotonicity?

- Are there any methods that are group rational and reasonable?

There are many different questions that have not been answered. This paper has not even begun to explore the possibilities, but hopefully it will give others some background and maybe spark some new ideas.

APPENDIX A

VALUE ALLOCATION: AN AXIOMATIC APPROACH

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1. VALUE ALLOCATION PROBLEMS

A *value allocation problem* is a pair (N, v) where $N = \{1, 2, \dots, n\}$ is the set of *individuals* and v is a real-valued function on the subsets of N , called the *value function*, which satisfies $v(\emptyset) = 0$ and *superadditivity*: $v(S \cup T) \geq v(S) + v(T)$ for all groups S and T satisfying $S \cap T = \emptyset$. We usually ascribe an economic interpretation to the value function: v(S) is the value the individuals in S can jointly share if they cooperate as a group. In this case, superaddivity has the natural interpretation that two disjoint groups can obtain at least as much working together as when then work independently. We shall often "abuse notation" by writing v(12) and v(i) rather than v({1, 2}) and v({i}).

Example 1. A hypothetical television game show has three contestants competitively answering Trivial Pursuit types of questions during the initial round to determine potential earnings in the next round. In the second round they are given the opportunity to bargain with one another to split the available prize money. Adam, Bob and Carrie are told that they can divide up to $900 among the three of them if they come to a mutually agreeable division. If the three cannot agree upon a division, any pair can try to divide smaller amounts: Adam and Bob can divide $180; Adam and Carrie can divide $360; and Bob and Carrie can divide $540. If no pair can agree upon a division, then each contestant receives no money. This situation can be modeled by a value allocation problem with three individuals and the value function $v(1) = v(2) = v(3) = 0$, $v(12) = 18$, $v(13) = 36$, $v(23) = 54$, and $v(123) = 90$. Here individuals 1, 2 and 3 can be identified with Adam, Bob and Carrie, respectively, and v(S) is interpreted as the amount of money (in tens of dollars) that the contestants in S can expect to divide if they and no one else cooperate as a group. Superadditivity holds because

$$v(1) + v(2) = 0 \leq 18 = v(12) \qquad v(12) + v(3) = 18 \leq 90 = v(123)$$
$$v(1) + v(3) = 0 \leq 36 = v(13) \qquad v(13) + v(2) = 36 \leq 90 = v(123)$$
$$v(2) + v(3) = 0 \leq 54 = v(23) \qquad v(23) + v(1) = 54 \leq 90 = v(123)$$

Example 2. Delphi, Eddytown and Franconia are the three cities in a rapidly growing county. New educational facilities will be required soon to accomodate the growing number of school-age children. Initially, each city estimated the cost to expand their own educational facilities: \$3.1, 3.4 and 4.6 million for Delphi, Eddytown and Franconia, respectively. Because of the large expense, the three cities decided to explore joint ventures with each other. It was determined that Delphi and Eddytown could meet their needs jointly for \$5.9 million; Delphi and Franconia could meet their needs jointly for \$5.3 million; Eddytown and Franconia could meet their needs jointly for \$5.6 million; and all three could meet their needs jointly for \$8.1 million. This situation can be modeled by a value allocation problem with three individuals and the value function $v(1) = v(2) = v(3) = 0$, $v(12) = 6$, $v(13) = v(23) = 24$, and $v(123) = 30$. Here individuals 1, 2 and 3 can be identified with Delphi, Eddytown and Franconia, respectively, and $v(S)$ is interpreted as the amount of money (in hundred-thousands of dollars) that the cities in S can expect to save if they and no one else cooperate as a group. For example, Delphi and Franconia save $31 + 46 - 53 = 22$.

Example 3. The value function may have a political power, rather than an economic, interpretation. A county board consists of four members. Because each board member represents a different town and the four towns in the county have different populations, a weighted voting scheme is used to make decisions. Specifically, the board members representing the towns of Anthrax, Babbage, Cleo, and Dodgeson have 5, 3, 2, and 1 votes, respectively. A motion passes if and only if it receives at least 6 votes. This situation can be modeled by a value allocation problem with four individuals and the value function $v(1) = v(2) = v(3) = v(4) = v(23) = v(24) = v(34) = 0$ and $v(12) = v(13) = v(14) = v(123) = v(124) = v(134) = v(234) = v(1234) = 0$. Here individuals 1, 2, 3 and 4 can be identified with the board members representing the towns of Anthrax, Babbage, Cleo, and Dodgeson, respectively, and $v(S) = 1$ if the board members in S can pass a motion while $v(S) = 0$ if the board members in S cannot pass a motion. In this situation, superaddivity implies that no pair of disjoint groups can both win simultaneously.

<u>Exercise 1.1</u>. Verify that the value functions defined in examples 2 & 3 are superadditive.

We are interested in those situations when all individuals cooperate. An *allocation* for the value allocation problem (N, v) is a vector $x = (x_1, x_2, \dots, x_n)$ satisfying the efficiency property: $\sum_{i=1}^{n} x_i = v(N)$. If x is an allocation for the value allocation problem (N, v), then we call x_i the *payoff* to individual i. The vectors $(60, 40, -10)$ and $(0, 0, 90)$ are allocations for the value allocation problem in example 1. Of course, neither one seems to make much sense in the context of example 1. We would like the allocation chosen for a value allocation problem to have the interpretation that these are the payoffs that either *would* or *should* occur in the context of the application. The allocation $(60, 40, -10)$ is not tenable because the third individual could obtain 0 on her own but is asked to pay 10. It would be irrational for the third individual to accept this allocation. An allocation x for (N, v) is called *individually rational* if the payoff to each individual i is at least $v(i)$. The set of all individually rational allocations is given by

$$IR(N, v) = \{ x \in \mathbb{R}^n : \sum_{i=1}^{n} x_i = v(N) \text{ and } x_i \geq v(i) \text{ for all } i \in N \}.$$

The allocation $(0, 0, 90)$ is individually rational, but the group $\{1, 2\}$ receives a zero payoff while it could obtain 18 by not cooperating with the third individual. It would seem irrational for the group $\{1, 2\}$ to accept this allocation. An allocation x for (N, v) is called *group rational* if the sum of the payoffs to any group S is at least $v(S)$. The set of all group rational allocations is given by

$$GR(N, v) = \{ x \in \mathbb{R}^n : \sum_{i \in N} x_i = v(N) \text{ and } \sum_{i \in S} x_i \geq v(S) \text{ for all } S \subset N \}.$$

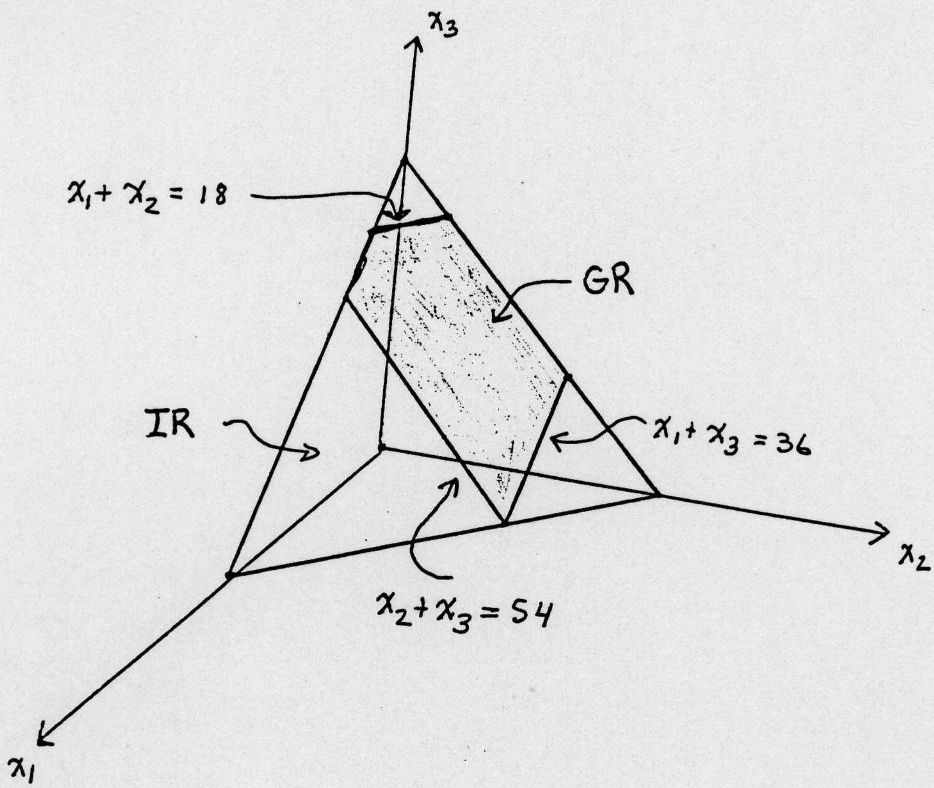
Consider example 1. For this value allocation problem, $IR(N, v) = \{ (x_1, x_2, x_3) : x_1 + x_2 + x_3 = 90 \text{ and } x_1, x_2, x_3 \geq 0 \}$ and $GR(N, v) = \{ (x_1, x_2, x_3) : x_1 + x_2 + x_3 = 90,\ x_1 + x_2 \geq 18,\ x_1 + +x_3 \geq 36,\ x_2 + x_3 \geq 54, \text{ and } x_1, x_2, x_3 \geq 0 \}$. These sets are represented geometrically in the figure on page 6.

Exercise 1.2. Find IR(N, v) and GR(N, v) for the value allocation problem in example 2. Show these sets geometrically.

Exercise 1.3. Consider the value allocation problem with three individuals and the value function $v(1) = v(2) = v(3) = 0$, $v(12) = v(13) = v(23) = 1$, and $v(123) = 2$. Find IR(N, v) and GR(N, v), and show these sets geometrically.

Exercise 1.4. Show that $IR(N, v) \neq \emptyset$ for all value allocation problems (N, v).

According to exercise 1.4, individually rational allocations always exist. Unfortunately, some value allocation problems have *no* group rational allocations. Consider example 3. If x is group rational, then $x_1 + x_2 \geq 1$, $x_1 + x_3 \geq 1$, $x_1 + x_4 \geq 1$, and $x_2 + x_3 + x_4 \geq 1$, which implies that $x_1 + x_2 + x_3 + x_4 \geq 5/2$ (multiply the first three inequalities by $1/3$, multilply the last by $2/3$, and add these inequalities together). But efficiency implies that $x_1 + x_2 + x_3 + x_4 = 1$. Since $1 < 5/2$, there cannot be a group rational allocation.


x_3
$x_1 + x_2 = 18$
GR
IR
$x_1 + x_3 = 36$
x_2
$x_2 + x_3 = 54$
x_1

2. ALLOCATION METHODS

An *allocation method* is a function that assigns to each value allocation problem an allocation. This section presents several allocation methods.

2.1 Basic Methods

Equal Allocation of Joint Value first allocates to each individual what they could obtain on their own, $v(i)$, and then the amount remaining, $v(N) - \sum_{i \in N} v(i)$, is divided equally among the individuals.

$$\theta_i^{EAJV}(N, v) = v(i) + \tfrac{1}{n}\left[v(N) - \sum_{i \in N} v(i)\right].$$

For Example 1, it is easy to calculate that $\theta^{EAJV}(N, v) = (30, 30, 30)$.

Equal Allocation of Nonseparable Value first allocates to each individual their "separable value," which is their marginal worth to the group of all individuals, and then the amount remaining is divided equally among the individuals.

$$\theta_i^{EANV}(N, v) = s_i + \tfrac{1}{n}\left[v(N) - \sum_{j \in N} s_j\right], \text{ where } s_j = v(N) - v(N-\{j\}).$$

For example 1, it is easy to calculate that $s = (36, 54, 72)$ and $\theta^{EANV}(N, v) = (12, 30, 48)$.

Proportional Allocation of Joint Value first allocates to each individual $v(i)$, as in EAJV, but the amount remaining is divided among the individuals in proportion to r_i, the remaining marginal value each adds to the group of all individuals.

$$\theta_i^{PAJV}(N, v) = v(i) + \frac{r_i}{\sum_{j \in N} r_j}\left[v(N) - \sum_{i \in N} v(i)\right], \text{ if } \sum_{j \in N} r_j \neq 0, \text{ and}$$

$$= v(i) + \frac{1}{n}\,[v(N) - \sum_{i \in N} v(i)], \text{ if } \sum_{j \in N} r_j = 0,$$

where $r_j = v(N) - v(N-\{j\}) - v(j)$. For example 1, it is easy to calculate that $r = (36, 54, 72)$ and $\theta^{EANV}(N, v) = (20, 30, 40)$.

Exercise 2.1. Find the EAJV, EANV, and PAJV allocations for examples 2 and 3.

2.2 The Shapley Value

Suppose the order in which each individual decides to cooperate is considered in making an allocation to each individual. Specifically, each individual could be allocated her marginal worth as she decides to cooperate. For example, in the ordering 1, 2, 3 of a three individual value allocation problem individual 1 would be allocated $v(1)$, individual 2 would be allocated $v(12) - v(1)$, and individual 3 would be allocated $v(123) - v(12)$. Now consider doing this for all possible orderings. For example 1, we obtain

MARGINAL VALUE CONTRIBUTED BY	ORDER INDIVIDUALS COOPERATE 123	132	213	231	312	321	SUM	ϕ_i SUM/6
INDIVIDUAL 1	0	0	18	36	36	36	126	21
INDIVIDUAL 2	18	54	0	0	54	54	180	30
INDIVIDUAL 3	72	36	72	54	0	0	234	39

The Shapley value for individual i is the average of the marginal values individual i brings to the group of all individuals over all possible individual orderings. The Shapley Value can also be written in the following closed-form:

$$\phi_i(N, v) = \sum_{S \subseteq N} \frac{(s-1)!\,(n-s)!}{n!}\,[v(S) - v(S-\{i\})], \text{ where } s \text{ denotes } |S|.$$

For example 1, we can calculate the Shapley value using the above formula as follows:

S	1	2	3	12	13	23	123	i	ϕ_i
$(s-1)!(n-s)!/n!$	1/3	1/3	1/3	1/6	1/6	1/6	1/3		
$v(S) - v(S-\{1\})$	0	0	0	18	36	0	36	1	21
$v(S) - v(S-\{2\})$	0	0	0	18	0	54	54	2	30
$v(S) - v(S-\{3\})$	0	0	0	0	36	54	72	3	39

Exercise 2.2. Find the Shapley Value for examples 2 and 3.

2.3 The Nucleolus and Related Methods

Given a value allocation problem (N, v), let $e(x, S) = v(S) - \sum_{i \in S} x_i$ be the excess of group S relative to the cost allocation x; this is a measure of how much group S is likely to complain about the allocation x, because $e(x, S)$ is the difference between what group S can obtain on its own and what it would obtain according to x. Let $e(x)$ be the vector of excesses $e(x, S)$, $S \neq \emptyset$, N, ordered from highest to lowest. We say that a vector y is lexicographically smaller than a vector z if there is a j for which $y_j < z_j$ and $y_i = z_i$ for all $i < j$, that is, in a dictionary of vectors, y would appear before z. The nucleolus is the individually rational allocation $\nu(N, v)$ that minimizes $e(x)$ lexicographically. In words, the nucleolus is the individually rational allocation which lexicographically minimizes the maximum excesses: the largest complaint is as small as possible and is voiced by as few groups as possible, the next largest complaint is as small as possible and is voiced by as few groups as possible, etc.

For example 1, if $x^1 = (50, 20, 20)$, then

S	1	2	3	12	13	23
e(x, S)	-50	-20	-20	-52	-34	14

and

$$e(x^1) = (14, -20, -20, -34, -50, -52).$$

Similarly,

$$x^2 = (20, 30, 40) \Rightarrow e(x^2) = (-16, -20, -24, -30, -32, -40)$$

$$x^3 = (18, 20, 52) \Rightarrow e(x^3) = (-18, -18, -20, -20, -34, -52)$$

$$x^4 = (18, 27, 45) \Rightarrow e(x^4) = (-18, -18, -27, -27, -27, -45).$$

Now $e(x^4)$ is lexicographically smaller than $e(x^3)$ which is lexicographically smaller than $e(x^2)$ which is lexicographically smaller than $e(x^1)$.

Is x^4 the nucleolus for this value allocation problem? Suppose x is the nucleolus. Since x^4 is an imputation, it follows that $e(x)$ is either equal to or lexicographically smaller than $e(x^4)$. In particular, no excess relative to x can be larger than -18. So, $e(x, 1) = 0 - x_1 \leq -18$ and $e(x, 23) = 54 - x_2 - x_3 \leq -18$ which simplify to $x_1 \geq 18$ and $x_2 + x_3 \geq 72$. Since $x_1 + x_2 + x_3 = 90$, it follows that the two inequalities must be equalities: $x_1 = 18$ and $x_2 + x_3 = 72$. It also follows that the excesses of $\{1\}$ and $\{2, 3\}$ relative to x must both be equal to -18. We can now use the fact that $e(x)$ is either equal to or lexicographically smaller than $e(x^4)$ once again to note that no remaining excess relative to x can be larger than -27. In particular, $e(x, 2) = 0 - x_2 \leq -27$ and $e(x, 13) = 36 - x_1 - x_3 \leq -27$ which simplify (using $x_1 = 18$) to $x_2 \geq 27$ and $x_3 \geq 45$. Since $x_2 + x_3 = 72$, it follows that the two inequalities must be equalities: $x_2 = 27$ and $x_3 = 45$. Thus, $x = x^4$ is the nucleolus.

The definition and example calculation above suggests the following iterative computational procedure. In this procedure, α_k is the k-th largest excess and $\mathcal{C}_k - \mathcal{C}_{k-1}$ is the set of groups on which the k-th largest excess is attained.

<u>Step 1</u>. Let $X_0 = \text{imp}(N, v)$, $\mathcal{C}_0 = \{\emptyset, N\}$, and $k = 1$.

<u>Step 2</u>. Minimize the maximum excess among groups whose excesses have not already been set, that is solve

$$\begin{aligned} \alpha_k = \min \quad & \alpha \\ \text{s.t.} \quad & e(x, S) \leq \alpha, \quad S \notin \mathcal{C}_{k-1} \\ & x \in X_{k-1}. \end{aligned}$$

<u>Step 3</u>. Let $X_k = \{ x \in X_{k-1} : (x, \alpha)$ is an optimal solution to the problem in step 2$\}$,

and $\mathcal{C}_k = \{ S \notin \mathcal{C}_{k-1} : e(x, S) = \alpha_k \text{ for all } x \in X_k \}$.

Step 4. If X_k contains a single vector, then this vector is the nucleolus. Otherwise, increment k and go to step 2.

For example 1, we begin by solving the following linear program:

$$
\begin{array}{llll}
\min \alpha & & & \\
\text{s.t. } 0 - x_1 & \leq \alpha & (1) & \\
0 \quad - x_2 & \leq \alpha & (2) & \\
0 \quad\quad - x_3 & \leq \alpha & (3) & \\
18 - x_1 - x_2 & \leq \alpha & (12) & e(x, S) \leq \alpha \\
36 - x_1 \quad - x_3 & \leq \alpha & (13) & \\
54 \quad - x_2 - x_3 & \leq \alpha & (23) & \\
x_1 + x_2 + x_3 & = 90 & (123) & \\
x_1, \; x_2, \; x_3 & \geq 0 & & x \in X_0
\end{array}
$$

We show one way of solving this linear program. Combine sets of inequalities such that equation (123) may be used as a substitution. In this case, we could add (1), (2), and (3); (1) and (23); (2) and (13); (3) and (12); or (12), (13) and (23). By doing this we want to solve for α to see which set of inequalities is the most constraining. These sets of inequalities yield the following constraints on α :

(1) (2) (3) $\Rightarrow -30 \leq \alpha$

(1) (23) $\Rightarrow -18 \leq \alpha$

(2) (13) $\Rightarrow -27 \leq \alpha$

(3) (12) $\Rightarrow -36 \leq \alpha$

(12) (13) (23) $\Rightarrow -24 \leq \alpha$

Because $\alpha \geq -18$ is the most constraining, we set $\alpha = -18$ and so inequalities (1) and (23) must hold as equalities. Using inequalities (12) and (13), we conclude that $\alpha_1 = -18$, $X_1 = \{ x : x_1 = 18,\ x_2 + x_3 = 72,\ x_2 \geq 18,\ x_3 \geq 36 \}$, and $\mathcal{C}_1 = \{1, 23\}$. Since X_1 contains more than one allocation, we now solve the following linear program:

$$
\begin{array}{llr}
\min \alpha & & \\
\text{s.t.}\ 0 - x_2 \leq \alpha & (2) & \\
0 - x_3 \leq \alpha & (3) & \\
18 - x_1 - x_2 \leq \alpha & (12) & e(x, S) \leq \alpha \\
36 - x_1 - x_3 \leq \alpha & (13) & \\
x_1 = 18 & (1) & \\
x_2 + x_3 = 72 & (23) & \\
x_2 \geq 18 & & x \in X_1 \\
x_3 \geq 36 & &
\end{array}
$$

Using equation (1) to simplify, we obtain

$$
\begin{array}{ll}
\min \alpha & \\
\text{s.t.}\ 0 - x_2 \leq \alpha & (2) \\
0 - x_3 \leq \alpha & (3) \\
0 - x_2 \leq \alpha & (12) \\
18 - x_3 \leq \alpha & (13) \\
x_1 = 18 & (1) \\
x_2 + x_3 = 72 & (23) \\
x_2 \geq 18 &
\end{array}
$$

$$x_3 \geq 36$$

Clearly, inequalities (12) and (13) combined with equation (23) yields the most binding constraint: $-27 \leq \alpha$. We set $\alpha = -27$ and note that the inequalities (12) and (13) must hold with equality. Hence, we find that $\alpha_2 = -27$, $X_2 = \{ (18, 27, 45) \}$, and $\mathcal{C}_2 - \mathcal{C}_1 = \{ 2, 12, 13 \}$. Since X_2 contains one allocation, it is the nucleolus: $\nu(N, v) = (18, 27, 45)$.

Exercise 2.3. Find the nucleolus for examples 2 and 3. Hint: For example 3, sum the excess inequalities associated with groups 12, 13, 14, and *two times* the inequality associated with the group 234.

Instead of considering the total complaint of each group, one might be interested in the average complaint per individual of each group. The per capita nucleolus is the individually rational allocation which lexicographically minimizes the maximum *per capita* excesses. More formally, the *per capita nucleolus*, denoted by $\nu^{PC}(N, v)$, is the individually rational allocation $\nu(N, v)$ that minimizes pce(x) lexicographically, where the per capita excesses are defined by $\text{pce}(x, S) = [v(S) - \sum_{i \in S} x_i]/|S|$. For example 1, we begin by solving the following linear program:

$$
\begin{array}{llll}
\min \alpha & & & \\
\text{s.t. } 0 - x_1 & \leq \alpha & (1) & \\
0 \quad - x_2 & \leq \alpha & (2) & \\
0 \quad\quad - x_3 & \leq \alpha & (3) & \\
18 - x_1 - x_2 & \leq 2\alpha & (12) & e(x, S) \leq \alpha \\
36 - x_1 \quad - x_3 & \leq 2\alpha & (13) & \\
54 \quad - x_2 - x_3 & \leq 2\alpha & (23) & \\
x_1 + x_2 + x_3 & = 90 & (123) & \\
x_1, \; x_2, \; x_3 & \geq 0 & & x \in X_0
\end{array}
$$

We find that $\alpha_1 = -12$, $X_1 = \{ (12, 30, 48) \}$, and $\mathcal{C}_1 = \{ 1, 12, 13, 23 \}$. Since X_1 contains one allocation, it is the per capita nucleolus: $\nu^{PC}(N, v) = (12, 30, 48)$.

Exercise 2.4. Find the per capita nucleolus for examples 2 and 3. See hint to exercise 2.3.

Exercise 2.5*. What happens when we do not restrict the minimizations to individually rational allocations? The *prenucleolus* is the allocation that lexicographically minimizes the maximum excess over the set of all allocations. Show that the prenucleolus and nucleolus yield the same allocations. The *per capita prenucleolus* is the allocation that lexicographically minimizes the maximum per capita excess over the set of all allocations. Show that the per capita prenucleolus and per capita nucleolus do *not* always yield the same allocations.

Exercise 2.6*. It has been suggested that instead of minimizing the maximum complaint (or per capita complaint), we should minimize the maximum *spread* of complaints. The spread could be measured by the range or standard deviation of the complaints. Explore one or more of these ideas by explicitly defining an allocation method and applying it to examples 1-3.

2.4 The Tau Value

The tau value is based on the ideas of maximum and minimum payoff entitlements. First, note that if an allocation x is group rational with respect to the value allocation problem (N, v), then the payoff to each individual is no more than her separable value: $x_i = v(N) - \sum_{j \in N-\{i\}} x_j \le v(N) - v(N-\{i\}) = s_i$. So, each individual should not expect anymore than her separable value. This now implies that a individual i in a group S should not be forced to accept any less than $v(S) - \sum_{j \in S-\{i\}} s_j$. Hence, individual i should not be forced to accept any less than the maximum of these quantities. With this in mind, we define the maximum and minimum, respectfully, entitlements for individual i in the value allocation problem (N, v) to be

$$M_i = v(N) - v(N-\{i\}), \text{ and}$$

$$m_i = \max \{ v(S) - \sum_{j \in S-\{i\}} M_j : i \in S \subseteq N \}.$$

The tau value is the imputation that yields a straight-line compromise between the maximum and minimum entitlement allocations:

$$\tau(N, v) = \lambda m + (1 - \lambda) M$$

where

$$\lambda = \frac{\sum_{i=1}^{n} M_i - v(N)}{\sum_{i=1}^{n} M_i - \sum_{i=1}^{n} m_i}.$$

For example 1, $M = (36, 54, 72)$ and $m = (0, 0, 0)$. So, $\lambda = 4/9$ and $\tau = (20, 30, 40)$.

<u>Exercise</u> <u>2.7</u>. Find the tau value for example 2.

In order for the words in the definition to make sense, (1) each individual's minimum entitlement should be no more than her maximum entitlement, and (2) the value available to the group of all individuals should lie between the sum of the minimum entitlements and the sum of the maximum entitlements. We call such value allocation problems quasibalanced. A value allocation problem (N, v) is *quasibalanced* if $m_i \leq M_i$ for all $i \in N$, and $\sum_{i=1}^{n} m_i \leq v(N) \leq \sum_{i=1}^{n} M_i$. The tau value has been extended to value allocation problems that are not quasibalanced, but this extension is beyond the scope of this manuscript.

<u>Exercise</u> <u>2.8</u>. Show that example 3 is not a quasibalanced value allocation problem.

<u>Exercise</u> <u>2.9*</u>. A different reasonable way to define the maximum entitlement would be as the maximum marginal value the individual adds to a group, that is, $M_i = \max \{ v(S) - v(S-\{i\}) : i \in S \subseteq N \}$. Similarly, the minimum entitlement could be defined as the minimum marginal value the individual adds to a group. Explore one or more of these ideas by explicitly defining an allocation method and applying it to examples 1-3.

3. PROPERTIES OF ALLOCATION METHODS

We would like the allocations chosen for a value allocation problem to have the interpretation that these are the payoffs that either *would* or *should* occur in the context of the application. The postive question, "What allocation would occur?" involves notions of bargaining and rationality. The normative question, "What allocation should occur?" involves notions of fairness and arbitration. In this section we define and examine a number of properties for allocation methods that capture some aspects of these notions of bargaining and fairness. We emphasize here that an allocation method will be said to possess a property if and only if the given condition holds on *all* value allocation problems.

3.1 Basic Properties

In this section, we describe a number of properties which any allocation method should possess. All of the methods described in section 2 possess these properties, except that PAJV is not continuous. At the other extreme, the following pathologic allocation method can be shown to possess none of the basic properties: $\theta_1(N, v) = v(N)^2$; $\theta_2(N, v) = v(N)$ if $v(2) > v(1)$; and $\theta_i(N, v) = 0$ if $i \neq 1$ or 2.

An allocation method satisfies the equal treatment property if two individuals receive the same payoff whenever they have the same effect on the value function: if $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq N - \{i, j\}$, then $\theta_i(N, v) = \theta_j(N, v)$.

An individual's payoff should not depend on her assigned number. An allocation method is symmetric if for all permutations π of N and for all individuals $i \in N$, it follows that $\theta_{\pi(i)}(N, \pi v) = \theta_i(N, v)$ where πv is the value function defined by $\pi v(\pi S) = v(S)$ for all $S \subseteq N$.

Exercise 3.1. Show that if an allocation method is symmetric, then it satisfies the equal treatment property. Exhibit an allocation method that is *not* symmetric but satisfies the equal treatment property.

It would seem like common sense for an allocation method not to depend on the units of value being used. A method is proportionate if for any $\alpha \neq 0$, it follows that $\theta(N, \alpha v) = \alpha\theta(N, v)$ where αv is the value function defined by $(\alpha v)(S) = \alpha v(S)$ for all $S \subseteq N$.

Suppose u is a value function and i is a individual satisfying $u(S) = v(S) + b$ for all groups S containing i, and $u(S) = v(S)$ for all groups S not containing i. The additional worth of groups with respect to u is due exclusively to individual i, and so she should receive that additional value, that is, $\theta_i(N, u) = \theta_i(N, v) + b$ and $\theta_j(N, v') = \theta_j(N, v)$ for $j \neq i$. A method that satisfies this property is said to be value separable.

It is sometimes convenient to refer to an allocation method that is both proportionate and value separable. Such a method is called covariant. An allocation method is covariant if for any $\alpha \neq 0$ and $b \in \mathbb{R}^n$, it follows that $\theta(N, u) = \alpha\theta(N, v) + b$ where u is defined by $u(S) = \alpha v(S) + \sum_{i \in S} b_i$ for all $S \subseteq N$.

Exercise 3.2. Show that an allocation method is covariant if and only if it is proportionate and value separable.

A small change in the value allocation problem should cause only a small change in the chosen allocation. An allocation method is continuous if whenever v^k and v are group functions satisfying $v^k(S) \rightarrow v(S)$ for all $S \subseteq N$, then $\theta_i(N, v^k) \rightarrow \theta_i(N, v)$ for all $i \in N$. The method PAJV is not continuous. Indeed, let $u^k(1) = u^k(2) = u^k(3) = 0$, $u^k(12) = u^k(13) = u^k(23) = 1 - 1/k$, and $u^k(123) = 1$, and let $v^k(1) = v^k(2) = v^k(3) = 0$, $v^k(12) = v^k(13) = 1 - 2/k$, $v^k(23) = 1 - 1/k$, and $v^k(123) = 1$. Then $\theta(N, u^k) = (1/3, 1/3, 1/3)$ and $\theta(N, v^k) = (2/5, 2/5, 1/5)$ although both sequences of allocation problems converge to the same allocation problem.

Exercise 3.3. Suppose θ satisfies the equal treatment property and is value separable. Show that if $N = \{1, 2\}$, then $\theta_i(N, v) = v(i) + \frac{1}{2}[v(12) - v(1) - v(2)]$ for $i = 1, 2$.

3.2 Rationality and Reasonableness

An allocation method is individually rational if no individual could ever do better on his own, that is, $\theta_i(N, v) \geq v(i)$ for all $i \in N$. If some individual could do better on his own, he might not be willing to cooperate with the rest of the individuals.

The Shapley value is individually rational on superadditive value allocation problems. Indeed, consider the marginal contribution of individual 1 given different orders for the individuals. When individual 1 enters first, his marginal contribution is $v(1)$. When individual 1 enters after a single other individual i, his marginal contribution is $v(i1) - v(i) \geq v(1)$ by superadditivity. In general, $v(S \cup \{1\}) - v(S) \geq v(1)$ by superadditivity. Since the Shapley Value is a weighted average of these marginal contributions, individual rationality follows. On the other hand, the per capita prenucleolus is *not* individually rational on all superadditive value allocation problems because this method yields the allocation $(-1, 7, 7, 7)$ for the 3-person value allocation problem $v(1) = v(2) = v(3) = v(4) = v(12) = v(13) = v(14) = 0$, $v(23) = v(24) = v(34) = v(123) = v(124) = v(134) = 16$, and $v(234) = v(1234) = 20$.

Rationality can be extended to groups instead of individual individuals. After all, if a group is better off on its own, then there would be no reason to cooperate with the rest of the individuals. This implies that an allocation should be in the core of the value allocation problem if the core is nonempty. Therefore, an allocation method is group rational if $\theta(N, v) \in \text{core}(N, v)$ whenever $\text{core}(N, v) \neq \emptyset$.

The Shapley value is *not* group rational. Indeed, consider the 3-person value allocation problem $v(1) = 1$, $v(2) = v(3) = 0$, $v(12) = 3$, $v(13) = v(23) = 4$, and $v(N) = 6$. The Shapley value for this value allocation problem is $(13/6, 11/6, 12/6)$. The core for this value allocation problem is nonempty because the allocation $(2, 2, 2) \in \text{core}(N, v)$; however, $\phi(N, v) \notin \text{core}(N, v)$ since $23/6 = \phi_2(N, v) + \phi_3(N, v) \leq v(23) = 4$. On the other hand, the nucleolus is group rational because if the core is nonempty, there exists an allocation with a nonpositive excess vector. Since the nucleolus is the imputation that minimizes the maximum excess, its excess vector will also have nonpositive components, and so the nucleolus is in the core.

We have seen that group rational allocations do not always exist. If the value allocation problem at hand has an empty core, then for any proposed allocation, there is a group which can do better on its own. So, there are always individuals with credible objections to any proposed allocation. An allocation x is called *stable* if for any objection there is a counterobjection. Formally, an objection of individual i against individual j with respect to the allocation x is an allocation y and a group S satisfying

(1) $i \in S$ and $j \notin S$,

(2) $y_k \geq x_k$ for all $k \in S$ and $y_i > x_i$, and

(3) $\sum_{k \in S} y_k \leq v(S)$ and $\sum_{k \in N-S} y_k \leq v(N-S)$.

A counterobjection of individual j against individual i with respect to the allocation x and objection (y, S) is an allocation z and a group T satisfying

(1) $j \in T$ and $i \notin T$,

(2) $z_k \geq x_k$ for all $k \in T$ and $z_k \geq y_k$ for all $k \in S \cap T$, and

(3) $\sum_{k \in T} z_k \leq v(T)$ and $\sum_{k \in N-T} z_k \leq v(N-T)$.

It is rational for individuals and groups to obtain as much when cooperating as they could obtain on their own. It is reasonable for individuals and groups to not receive more than their maximum marginal contribution. Given a value allocation problem (N, v), a individual $i \in N$, and groups $T \subseteq S \subseteq N$, individual i makes a marginal contribution to group S of $v(S) - v(S - \{i\})$, and group T makes a marginal contribution to group S of $v(S) - v(S-T)$. In both cases we are interested in what the value of the group S is without a particular individual or group of individuals or what that particular individual or group of individuals has contributed to the value of S. A method is individually reasonable if no individual receives more than her maximum marginal contribution to any group $S \subseteq N$, that is, $\theta_i(N, v) \leq \max\{ v(S) - v(S-\{i\}) : i \in S \subseteq N \}$ for all $i \in N$. A method is group reasonable if no group receives more that their maximum marginal contribution, that is, $\sum_{i \in T} \theta_i(N, v) \leq \max\{ v(S) - v(S-T) : T \in S \subseteq N \}$ for all $T \subseteq N$ whenever such an allocation exists.

The Shapley value is individually reasonable because it considers all possible marginal contributions for each specific individual and then takes the average of these values. Therefore, there is no way that the Shapley value could ever be larger than the maximum marginal contribution for that individual.

3.3 Consistency Properties

If a individual's marginal contribution to any group is only what they could make on their own, then in all fairness that individual should receive only this amount. This individual is often referred to as a "dummy" individual. In other words, if $i \in N$ satisfies $v(S \cup \{i\}) = v(S) + v(i)$ for all groups S that do not contain i, then $\theta_i(N, v) = v(i)$. Allocation methods which possess this property are said to be individually subsidy free.

An extension to individually subsidy free is the property of group subsidy free. If a given group is such that each subgroup's marginal contribution to any group of individuals not in the given group is what the subgroup can make on its own, then the given group should receive its value. In other words, if $T \subseteq N$ satisfies $v(S) = v(S \cap T) + v(S - T)$ for all groups $S \subseteq N$, then $\sum_{i \in T} \theta_i(N, v) = v(T)$.

When $T \subseteq N$ satisfies $v(S) = v(S \cap T) + v(S - T)$ for all groups $S \subseteq N$, it is as if the individuals in T and the individuals in $N - T$ are playing two completely independent value allocation problems. Of course, there should be no cross subsidies in the combined value allocation problem (as required by the group subsidy free property), but it also seems reasonable that the allocation should not depend upon whether the combined or two separate value allocation problems are being played. An allocation method is individually separable if whenever $T \subseteq N$ satisfies $v(S) = v(S \cap T) + v(S - T)$ for all groups $S \subseteq N$, it follows that $\theta_i(N, v) = \theta_i(T, v_T)$ for all $i \in T$, where v_T is defined by $v_T(S) = v(S)$ for all $S \subseteq T$.

As motivated above, the property of individual separability says that what happens in a value allocation problem should be replicated when the value allocation problem is restricted to a subset of its individuals. The last two properties that we consider in this section are stronger versions of this idea. An allocation method is Sobolev consistent if $\theta_i(N, v_{T,\,\theta(N,\,v)}) = \theta_i(N, v)$ for all $i \in T$ and $T \subseteq N$, where $(T, v_{T,\,x})$ is the reduced value allocation problem defined by

$$v_{T,\,x}(T) = \sum_{i \in T} x_i\,, \text{ and}$$
$$v_{T,\,x}(S) = \max_{R \subseteq N-T} \{ v(S \cup R) - \sum_{i \in R} x_i \} \text{ for } S \subset T\,.$$

An allocation method is Hart & Mas-Colell consistent if $\theta_i(N, v_{T,\,\theta(N,\,v)}) = \theta_i(N, v)$ for all $i \in T$ and $T \subseteq N$, where $(T, v_{T,\,x})$ is the reduced value allocation problem defined by

$$v_{T,\,x}(S) = v(S \cup (N-T)) - \sum_{i \in N-T} x_i \text{ for } S \subseteq T\,.$$

Exercise 3.4. Show that if an allocation method is individually separable, then it is group subsidy free. Show that if an allocation method is group subsidy free, then it is individually subsidy free.

3.4 Monotonicity and Additivity Properties

Monotonicity considers changes made to value allocation problems and places reasonable restrictions on the allocations as a result of these changes.

Suppose we have a value allocation problem (N, v) and increase the value of the group of all players, $v(N)$. An allocation method is aggregate monotone if there is no decrease in the allocation to any individual $i \in N$, which seems natural. If we call the original value allocation problem (N, v^1) and the changed value allocation problem (N, v^2) this property says that if $v^1(N) \leq v^2(N)$ and $v^1(S) = v^2(S)$ for $S \neq N$, then $\theta_i(N, v^1) \leq \theta_i(N, v^2)$ for all $i \in N$. The Shapley value is aggregate monotone because $v^1(N) - v^1(N\text{-}\{i\}) \leq v^2(N) - v^2(N\text{-}\{i\})$ for all $i \in N$, therefore $\phi(N, v^1) \leq$

$\phi(N, v^2)$.

Group monotonotonicity is very similar to aggregate monotonicity, but instead of examining increases in just the group of all individuals, it considers changes to any group and restricts the new allocation for any individual in these groups. Specifically, for every fixed $T \subseteq N$, if $v^1(T) \leq v^2(T)$ and $v^1(S) = v^2(S)$ for $S \neq T$, then $\theta_i(N, v^1) \leq \theta_i(N, v^2)$ for all $i \in T$. Equivalently, for every fixed $i \in N$, if $v^1(S) \leq v^2(S)$ for all S containing i, and $v^1(S) = v^2(S)$ for all S not containing i, then $\theta_i(N, v^1) \leq \theta_i(N, v^2)$.

Strict Monotonicity is the same as group monotonicity except we have strict inequalities. In other words, for every fixed $T \subseteq N$, if $v^1(T) < v^2(T)$ and $v^1(S) = v^2(S)$ for $S \neq T$, then $\theta_i(N, v^1) < \theta_i(N, v^2)$ for all $i \in T$.

The notion of strong monotonicity concentrates on marginal contributions instead of increases in value. For this property if for some individual $i \in N$ and value allocation problems (N, v^1) and (N, v^2), individual i's marginal contribution to each group S in value allocation problem v^1 is less than or equal to its marginal contribution to each group in value allocation problem v^2 then individual i should be allocatied at least as much in v^2 as in v^1. So, for every fixed $i \in N$, if $v^1(S) - v^1(S-\{i\}) \leq v^2(S) - v^2(S-\{i\})$ for all S containing i, then $\theta_i(N, v^1) \leq \theta_i(N, v^2)$.

Oftentimes, accountants like to breakdown overall costs into their component parts. For example, the cost of a municipal project might consist of capital and maintenance costs. Surely, the allocation method should give identical results whether one considers the separate costs independently and then adds the imputed allocations or considers the costs jointly to arrive at an allocation. Consider two value allocation problems (N, v^1) and (N, v^2) and a third value allocation problem $(N, v^3) = (N, v^1 + v^2)$. A method is said to be additive if the allocations made for v^1 and v^2 sum or add up to the allocation in v^3. In other words, $\theta(N, v^1 + v^2) = \theta(N, v^1) + \theta(N, v^2)$ where $(N, v^1 + v^2)(S) \equiv v^1(S) + v^2(S)$ for all $S \subseteq N$.

Exercise 3.5. Show that if an allocation method is strong monotone or strict monotone, then it is group monotone. Show that if an allocation method is group monotone, then it is aggregate monotone.

4. SOME IMPORTANT THEOREMS

Theorem 1 (Shapley, 1953). The Shapley value is the unique value allocation method that is symmetric, individually subsidy free and additive.

Theorem 2 (Young, 1985). The Shapley value is the unique allocation method that is symmetric and strongly monotonic.

Theorem 3 (Young, 1985 and Jew and Housman, 1989). There exists no allocation method that is both group rational and group monotone.

Theorem 4 (Sobolev, 1975). The nucleolus is the unique allocation method that is symmetric, covariant and Sobolev consistent.

Theorem 5 (Hart & Mas-colell, 1987). The Shapley value is the unique allocation method that is symmetric, covariant and Hart & Mas-colell consistent.

Exercise 4.1*. For each allocation method M and property P considered, show whether M satisfies P on the class of superadditive value allocation problems. You may use theorems 1 - 5 given above in your proofs.

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APPENDIX: RELATIONSHIP TO GAME THEORY

Term Used in This Paper	Term Used in the Game Theory Literature
value allocation problem	superadditive cooperative game, superadditive game in coalitional function form
individual	player
group	coalition
allocation	preimputation, payoff vector
individually rational allocations	imputations
group rational allocations	core payoff vectors
covariant	relative invariance under strategic equivalence
individually subsidy free	dummy property, dummy axiom

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