Some observations on a notion of consistency

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0 Introduction

Reduced game concepts model renegotiation, within subsets of the players, of the payoffs given by a particular value. A value is consistent when any reallocation awards players exactly their payoff in the original game. One of our eventual goals is to find conditions on a reduced game concept which would guarantee the existence of a value consistent with respect to the reduced game concept. Here we explore some techniques possibly useful in that endeavor.

This paper arose out of an earlier, erroneous paper of mine, which claimed to prove an applicable result. This paper makes no such claim; first, I have here left a key conjecture unproven (though I believe it is true); more importantly, one of the assumptions is so strict that probably any theorem is true only vacuously. The purpose of this paper is just to "see how far we can really get" within the framework of my ideas over the summer. Thanks to David Housman for finding the error. Apologies to all whom I misled.

1 Basic Notions

A game is a pair (N,v) where $N=\{1,2,\cdots,n\}$ represents the players and $v: 2^N \to \mathbb{R}$ with $v(\emptyset)=0$ represents the worths of coalitions of players. A value φ associates with each (N,v) a vector $\varphi(N,v)\in\mathbb{R}^n$ where $\varphi_i(N,v)$ is the payoff to player i.

Example: The Shapley value Sh is defined by

$$Sh_i(N,v)=\sum_{S\subseteq N}\frac{(n-s)!(s-1)!}{n!}\left[v(S)-v(S\backslash i)\right]$$

A reduced game concept RG associates with each 4-tuple (N,v,S,φ), where $S\subseteq N$, a new game $RG(N,v,S,\varphi)$ on player set S. Since we will encounter no serious ambiguity regarding RG and N, this new game will be denoted by either (S,v_S^{φ}) or (S,v_S), usually the latter.

Examples: The Hart/Mas-Colell reduced game concept is defined by

$$v_S(T)=v(T\cup(N\backslash S))-x(N\backslash S) \text{ where } x=\varphi\left(T\cup(N\backslash S), v|_{2^{T\cup(N\backslash S)}}\right)$$

The Davis/Maschler reduced game concept is defined by

$v_S(S) = x(S)$, where $x = \varphi(N,v)$

$v_S(T) = \max_{R \subseteq N \setminus S} (v(T \cup R) - x(R))$ for $T \subset S$

2 Definitions

Notation: For $x \in \mathbb{R}^n$, $x(S) := \sum_{i \in S} x_i$

Identify $\{i\}$ and i.

Write I_k for $\{1, 2, \cdots, k\}$.

Definitions: A game (N,v) is

0-monotonic if $\forall i \in N, \forall S \subseteq N \setminus i$, we have $v(S \cup i) \geq v(S) + v(i)$

superadditive if $\forall S, T \subseteq N$ with $S \cap T = \emptyset$, we have $v(S \cup T) \geq v(S) + v(T)$

convex if $\forall i \in N, \forall S \subseteq T \subseteq N \setminus i$, we have $v(S \cup i) - v(S) \leq v(T \cup i) - v(T)$

or, equivalently, if $\forall S, T \subseteq N$, we have $v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$

Definition: We say that $i \in N$ is a *dummy* in (N,v) if $\forall S \subseteq N, v(S \cup i) = v(S)$

Definitions: A value φ satisfies

DPP (dummy player property) if $\varphi_i(N,v) = 0$ whenever i is a dummy in (N,v)

SYM (symmetry) if for any permutation τ on N we have $\tau(\varphi(v)) = \varphi(\tau(v))$ where for $x \in \mathbb{R}^n$, $\tau(x)_i := x_{\tau(i)}$, and for $S \subseteq N, \tau(v)(S) := v(\tau(S))$

COV (covariance under strategic equivalence) if $\alpha\{> 0, \beta \in \mathbb{R}^n$ so that $\forall S \subseteq N$: $w(S) = \alpha v(S) + \beta(S)\} \Rightarrow \varphi(N,w) = \alpha\varphi(N,v) + \beta$. If the bracketed condition holds, then v and w are said to be *strategically equivalent.*

IR (individual rationality) at (N,v) if $\forall i \in N$, we have $\varphi_i(N,v) \geq v(i)$

EFF (efficiency) if $\sum_{i \in N} \varphi_i(N,v) = v(N)$

consistency, with respect to a given reduced game concept, if $\forall S \subseteq N, \forall i \in S$, $\varphi_i(S, v_S) = \varphi_i(N, v)$.

Definition: If a reduced game concept is continuous as a function on values when other parameters are fixed, then we will just say that it is *continuous* (with respect to given topologies on the relevant spaces).

Definition: If P is one of 0-monotonicity, superadditivity, or convexity, and (S, v_S) satisfies P whenever (N, v) does, then the reduced game concept is said to *preserve P*.

Definitions: A reduced game concept is said to *preserve*

dummies if $i \in S$ is a dummy in (S, v_S^φ) whenever it is a dummy in (N, v) and φ has DPP;

strategic equivalence if (S, v_S^φ) and (S, w_S^φ) are strategically equivalent whenever (N, v) and (N, w) are strategically equivalent and φ is covariant;

symmetry if for any permutation τ on N, any symmetric value φ, we have $\tau\left(v^\varphi_{\tau(S)}\right) = \tau(v)^\varphi_S$.

Definition: A reduced game concept is *regular* if $v_N = v$ and $\forall i \in N$: $v_S(i) \geq v(i)$

Definition: A regular reduced game concept is *contractive* if for all games (N, v) and (N, w) where $v(N) \leq w(N)$, all values φ satisfying SYM, COV, EFF, IR, and DPP, all $S \subseteq N$, we have the following:

$$2\rho\left(v_S^\varphi, (\overline{w})_S^\varphi\right) + w(N) - v(N) \leq 2\rho(v, w)$$

where $\rho(v, w) = \max_{T \subseteq N} |v(T) - w(T)|$ and where $\overline{w} \equiv w\frac{v(N)}{w(N)}$.

Remark: I have found no reduced game concept which satisfies this. Also, the proof below will require only that the inequality above hold for $\varphi \in \Phi$ and games in M_n (see **5** for notation), but since likely no reduced game concept satisifies even those requirements, I have chosen to make the definition concise.

3 Tau-consistency

With respect to a given reduced game concept, for any set Ω of efficient allocation methods defined on all games of size $\leq s$ for some fixed s, define mappings $\mathcal{L}$ and $\mathcal{M}$ from Ω to Ω by

$$\mathcal{L}(\varphi)_i(N,v) = \min_{S \ni i} \varphi_i(S, v_S) \text{ and } \mathcal{M}(\varphi)_i(N,v) = \max_{S \ni i} \varphi_i(S, v_S).$$

Define $\mathcal{T}: \Omega \to \Omega$ by $\mathcal{T}(\varphi)(N,v) = t(N,v)\mathcal{L}(\varphi)(N,v) + (1 - t(N,v))\mathcal{M}(\varphi)(N,v)$ where $t(N,v)$ is whatever $t \in [0,1]$, is such that $t\mathcal{L}(\varphi) + (1-t)\mathcal{M}(\varphi)$ is efficient. Note that $\mathcal{T}$ is well-defined because $\varphi(N,v)$ is on the hyperplane of efficient allocations, while $\mathcal{L}(\varphi)_i(N,v) \leq \varphi_i(N,v) \leq \mathcal{M}(\varphi)_i(N,v)$ for every i, so that $\mathcal{L}(\varphi)(N,v)$ and $\mathcal{M}(\varphi)(N,v)$ are on different sides of this hyperplane (or one of them is on the hyperplane); thus some unique point on the line segment through them is on the hyperplane.

Say that φ is *tau-consistent* with respect to a reduced game concept if $\mathcal{T}(\varphi) \equiv \varphi$. Note that consistency implies tau-consistency.

4 Topology

We will use the following:

Theorem (Ascoli): Let X be a compact space; consider the space $\mathfrak{C}(X, \mathbb{R}^n)$ of continuous functions $X \to \mathbb{R}^n$ in the sup metric ρ. A subset $\mathfrak{F}$ of $\mathfrak{C}(X, \mathbb{R}^n)$ is compact if and only if it is closed, bounded, and equicontinuous.

Theorem (Schauder): Any continuous mapping of a nonempty, convex, compact Banach space to itself has a fixed point.

5 Existence of a tau-consistent value

Theorem: Assume that a reduced game concept ($v \mapsto v_S$) preserves 0-monotonicity (resp. superadditivity or convexity), symmetry, dummies, and strategic equivalence; is continuous, regular, and contractive. Then for all n there exists a value defined on all 0-monotonic (resp. superadditive or convex) games of n or fewer players. This value is tau-consistent with respect to the reduced game concept, individually rational, symmetric, covariant, continuous, and satisfies dummy player property.

Proof: We prove for reduced game concepts which preserve 0-monotonicity. View a characteristic function on k players as an element of $\mathbb{R}^{2^k}$. For any positive integer n, let

$G_n = \{(I_k, [0,1]^{2^k}) \mid k = 1 \cdots n\}$. Let $M_n = \{g \in G_n \mid g$ is 0-monotonic$\}$. Denote the allocation space by $A_n = [0,1]^n$. Represent allocations to fewer than the full number of players by padding with zeros. Give A_n and $[0,1]^{2^k}$ the usual topologies, which are induced by the square metric $\rho(x,y) = \max |x_i - y_i|$. Give G_n the topology of the disjoint union of the $[0,1]^{2^k}$. Since the limit of 0-monotonic games is 0-monotonic, M_n is closed in a compact space, and is hence compact.

Let $\Phi = \{\varphi: M_n \rightarrow A_n \mid \varphi$ satisfies DPP, SYM, COV, EFF, IR, and

$$\forall k, \forall g_1, g_2 \in (I_k, [0,1]^{2^k}): \rho(\varphi(g_1), \varphi(g_2)) \leq 2\rho(g_1, g_2)\}.$$

Give Φ the uniform topology, induced by the metric $d(\varphi_1, \varphi_2) = \max_g \rho(\varphi_1(g), \varphi_2(g))$.

Since $Sh \in \Phi$, it is nonempty. Clearly it is convex. Since it is closed, bounded, and equicontinuous, by Ascoli's Theorem it is compact.

Claim: $\mathcal{T}$ maps Φ to itself.

Proof: We have that $\mathcal{T}(\varphi)$ satisfies

DPP because any dummy i in (N,v) is a dummy in (S, v_S) so $\mathcal{L}(\varphi)_i(N,v) = \min 0 = 0$ and $\mathcal{M}(\varphi)_i(N,v) = \max 0 = 0$, so $\mathcal{T}(\varphi)_i(N,v) = 0$.

COV because if $w = \alpha v + \beta$ then

$\mathcal{L}(\varphi)_i(N,w) = \min \varphi_i(S, (\alpha v + \beta)_S) = \min \varphi_i(S, \alpha v_S + \beta) = \min (\alpha\varphi_i(S, v_S) + \beta_i) = \alpha\mathcal{L}(\varphi)_i(N,v) + \beta_i$.

Similarly, $\mathcal{M}(\varphi)_i(N,w) = \alpha\mathcal{M}(\varphi)_i(N,v) + \beta_i$.

So $\sum_{i \in N} (t(N,v)\mathcal{L}(\varphi)_i(N,w) + (1 - t(N,v))\mathcal{M}(\varphi)_i(N,w)) = \alpha v(N) + \beta(N) = w(N)$.

It follows that $t(N,v) = t(N,w)$ and hence that $\mathcal{T}(\varphi)_i(N,w) = \alpha\mathcal{T}(\varphi)_i(N,v) + \beta_i$.

IR because for all S, $v_S(i) \geq v(i) \Rightarrow \varphi_i(S, v_S) \geq v(i)$. So $\max_S \varphi_i(S, v_S) \geq \min_S \varphi_i(S, v_S) \geq v(i)$ $\Rightarrow \mathcal{T}(\varphi)_i(N,w) \geq v(i)$.

SYM because $\forall i \in N, \forall \tau$ a permutation on N:

$\mathcal{L}(\varphi)_{\tau(i)}(v) = \min_{S \ni \tau(i)} \varphi_{\tau(i)}(v_S) = \min_{S \ni \tau(i)} \varphi_i(\tau(v_S)) = \min_{S \ni i} \varphi_i(\tau(v_{\tau(S)})) = \min_{S \ni i} \varphi_i(\tau(v)_S) = \mathcal{L}(\varphi)_i(\tau(v))$.

and similarly for $\mathcal{M}$ and hence for $\mathcal{T}$.

the Lipschitz condition: for any i, for any $g_1 = (N,v)$, $g_2 = (N,w)$, where $v(N) \le w(N)$,

$$\left|\mathcal{T}(\varphi)_i(g_1) - \mathcal{T}(\varphi)_i(g_2)\right| \le \left|\mathcal{T}(\varphi)_i(g_1) - \mathcal{T}(\varphi)_i(\overline{g}_2)\right| + \left|\mathcal{T}(\varphi)_i(\overline{g}_2) - \mathcal{T}(\varphi)_i(g_2)\right| \text{ where } \overline{g}_2 \equiv g_2\frac{v(N)}{w(N)}.$$

Conjecture: We have now

$$\left|\mathcal{T}(\varphi)_i(g_1) - \mathcal{T}(\varphi)_i(\overline{g}_2)\right| \le \max\left\{\max_i \left|\mathcal{L}(\varphi)_i(g_1) - \mathcal{L}(\varphi)_i(\overline{g}_2)\right|,\ \max_i \left|\mathcal{M}(\varphi)_i(g_1) - \mathcal{M}(\varphi)_i(\overline{g}_2)\right|\right\}.$$

I am quite sure that this or something very similar is true. Relevant facts are that $g_1(N) = \overline{g}_2(N)$ (so that g_1 and $\overline{g}_2$ share the same hyperplane of efficient allocations), and that $\mathcal{L}(\varphi)_i(g) \le \varphi_i(g) \le \mathcal{M}(\varphi)_i(g)$ for all i (so that the line segment from $\mathcal{L}(\varphi)(g)$ to $\mathcal{M}(\varphi)(g)$ cannot make too acute an angle with the hyperplane). A diagram for the two dimensional case makes this clear. □

Assume the conjecture true. Then

$$\left|\mathcal{T}(\varphi)_i(g_1) - \mathcal{T}(\varphi)_i(\overline{g}_2)\right| \le \left|\mathcal{L}(\varphi)_j(g_1) - \mathcal{L}(\varphi)_j(\overline{g}_2)\right|, \text{ say} \qquad \text{(proof for } \mathcal{M} \text{ would be similar)}$$

$$= \left|\min_S \varphi_j(S, v_S) - \min_S \varphi_j(S, \overline{w}_S)\right| \le \max_S \left|\varphi_j(S, v_S) - \varphi_j(S, \overline{w}_S)\right| \le 2 \max_S \rho(v_S, \overline{w}_S)$$

Also, $\left|\mathcal{T}(\varphi)_i(\overline{g}_2) - \mathcal{T}(\varphi)_i(g_2)\right| = \left(1 - \frac{v(N)}{w(N)}\right)\mathcal{T}(\varphi)_i(g_2) \le w(N) - v(N)$. So using contractiveness,

$$\left|\mathcal{T}(\varphi)_i(g_1) - \mathcal{T}(\varphi)_i(g_2)\right| \le 2 \max_S \rho(v_S, \overline{w}_S) + w(N) - v(N) \le 2\rho(g_1, g_2).$$ □

Claim: $\mathcal{T}$ is continuous

Proof: Pick any $\varphi \in \Phi$, any $\epsilon > 0$. For each $S \subseteq N$, the continuity of the reduced game concept and of any element in Φ implies that $\exists\ \delta_S > 0$ s.t. $d(\varphi, \theta) < \delta_S \Rightarrow d\left(\varphi_i(S, v_S^{\varphi}),\ \theta_i\left(S, v_S^{\theta}\right)\right) < \epsilon$. Let $\delta = \min \delta_S$. Then $\forall \theta \in \Phi$ with $d(\varphi, \theta) < \delta$ we have

$$d(\mathcal{L}(\varphi), \mathcal{L}(\theta)) = d\left(\min \varphi_i(S, v_S^{\varphi}),\ \min \theta_i\left(S, v_S^{\theta}\right)\right) \le \max d\left(\varphi_i(S, v_S^{\varphi})\ ,\ \theta_i\left(S, v_S^{\theta}\right)\right) \le \epsilon.$$

So $\mathcal{L}$ is continuous. Similarly, $\mathcal{M}$ is continuous. Since $t(\cdot)$ as a function on Φ is continuous, it follows that $\mathcal{T}$ is continuous. □

By Schauder's Fixed Point Theorem, $\mathcal{T}: \Phi \to \Phi$ has a fixed point φ^*. This value φ^* is by definition tau-consistent.

The fact that $\varphi^* \in \Phi$ yields the other desired properties.

Extend φ^* to a value ϕ on all 0-monotonic games by $\phi(N,v) := \varphi^*\left(N, \frac{v}{v(N)}\right) \cdot v(N)$, and note that ϕ retains all the desired properties. □

6 Open Questions

There is the matter of the conjecture; but beyond that:

First, how can the assumptions be weakened? The contractiveness requirement on the reduced game concept is clearly unacceptable. It has been difficult to find reasonable conditions which guarantee that $\mathcal{T}$ preserves the Lipschitz condition. If that is ever solved, another restriction to weaken is the preservation of 0-monotonicity, etc.

Second, how can the conclusions be strengthened? Clearly we want to be able to say something about consistency and not just the awkward tau-consistency (although I would suggest that it's not much more awkward than the tau value). After that, a natural next step is to guarantee existence of a value defined on games of arbitrary size.

Third, is there a "dual" theorem which, given a value satisfying certain conditions, guarantees existence of a reduced game concept with respect to which the value is consistent? The idea of the proof would be to take some given value, and look for fixed points of some mapping of reduced game concepts to reduced game concepts.

Fourth, is there an easier proof, using fixed-point methods, of Sobolev's characterization of the nucleolus? If a mapping similar to $\mathcal{T}$ were a contraction mapping, then existence and uniqueness of a consistent value would follow.

The intuitive similarity between consistency in cooperative game theory and equilibria in economic theory suggests that the fixed point approach so fruitful in the latter theory should be successful in the former.

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