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The Shapley Value and Partially Defined Games

by Jennifer Rich

Mathematics Department, Drew University

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Introduction

Many mathematicians are realizing the usefulness of cooperative game theory in the process of allocating costs or benefits among the participants of joint endeavors. Realizing that in reality, the determination of all coalitional worths may be prohibitively expensive or impractical, this paper is aimed at providing possible allocation methods for games which have unknown coalitional worths.

Definitions

In this paper, all games in which some coalitional worths are not known are referred to as partially defined games, or PDG's. Each game consists of a set of players, N, with $N = \{1, 2, ..., n\}$. Let M be a subset of N such that $1, n \in M$. A partially defined game with respect to M, or an *M-game*, is a real valued function ω on $\{S \subseteq N: |S| \in M\}$. The real number $\omega(S)$ is often called the worth of the coalition S. By defining M so that it always contains 1 and n, we insure that the worth of the singleton and grand coalitions are known, in addition to the worths of any other coalition whose size is in the set M.

An allocation for the M-game is a vector of payoffs $x \in \mathbb{R}^n$. An allocation method is a function from a class of games to allocations $\mathbb{R}^n$ which attempts to fairly distribute the costs or benefits of the joint venture. The

Shapley value φ is an allocation on N-games defined by the formula

$$\varphi_i(\omega)=\sum_{S\subseteq N,\, i\in S}\frac{(s-1)!\,(n-s)!}{n!}\,[\omega(S)-\omega(S-\{i\})]$$

where $s=|S|$ and $n=|N|$. This formula tells us that the Shapley value to player i in game w is player i's average marginal contribution over all possible orderings of players. An equivalent formula obtained through algebraic manipulation is

$$\varphi_i(\omega)=\frac{1}{n}\sum_{|S|\in N}\,[\sum_{|S|=s,\, i\in S}\binom{n-1}{s-1}^{-1}\omega(S)-\sum_{|S|=s,\, i\notin S}\binom{n-1}{s}^{-1}\omega(S)]$$

This form implies that player i should obtain an average over coalition sizes of the average worth of coalitions containing i minus the average worth of those not containing i. This becomes relevant when studying certain classes of M-games.

When dealing with a PDG, the unknown worths of coalitions can be estimated based on the class of game being considered. If Ω is a class of N-games, then the N-game $\hat{\omega}$ is called a Ω-extension of the M-game ω if $\hat{\omega} \in \Omega$ and $\hat{\omega}(S)=\omega(S)$ for all $|S| \in M$. In this paper, the set of all Ω-extensions of ω will be denoted ext(ω). Once the set of all Ω-extensions has been characterized it becomes possible to find a central Ω-extension of the M-game ω. Assuming a uniform probability distribution of the extensions, this paper examines the use of two different central extensions. First, the geometric center, denoted by centroid ext(ω) = $\mathbf{C}$, is used as a reasonable estimate of the underlying N-game. Second, the coordinate center of the ext(ω) is used as another means of approximating the central extension of the M-game. The central N-game, defined by either the centroid or the coordinate center, then becomes a sensible extension on which to apply the Shapley value.

The *zero-normalization* of an M-game ω is the M-game w defined by $w(S) = \omega(S) - \sum_{i\in S}\omega(\{i\})$ for all $|S| \in M$. This forces the worth of all the singleton coalitions to be equal to zero, without significantly changing the M-game. The zero-normalized extension w produces a shift in the worths of ext(ω) by the sum of the worths of the singleton coalitions. If the class of games is closed with respect to these singleton shifts (i.e. $\omega \in \Omega$ and $b \in \mathbb{R}^n$ then $\omega+b \in \Omega$ where $(\omega+b)(S) \equiv \omega(S)+\sum_{i\in S}b_i$ for all $S \subseteq N$), then the zero-normalization of the extensions of an M-game ω are the extensions of w, the zero-normalization of ω. In such cases, it follows that $\varphi_i(\omega) = w(\{i\}) + \varphi_i(w)$. All classes of games treated in this paper are closed with respect to singleton shifts; therefore, this paper will only consider M-games in which the worths of all singleton coalitions are equal to zero, since any given game can be zero-normalized and the final allocation adjusted accordingly.

This paper examines allocations for three classes of games. The N-game ω (defined by ext(ω) in an M-game) is

(1) *size zero-monotonic* if $\omega(S) - \sum_{i\in S}\omega(\{i\}) \le \omega(T) - \sum_{i\in T}\omega(\{i\})$ for all $S,T \subseteq N$ satisfying $|S| < |T|$;

(2) *zero-monotonic* if $\omega(S) + \omega(\{i\}) \le \omega(S \cup \{i\})$ for all $S \subseteq N$ and $i \in N-S$;

(3) *superadditive* if $\omega(S) + \omega(T) \le \omega(S \cup T)$ for all $S,T \subseteq N$ satisfying $S \cap T = \emptyset$.

Note that superadditive games and size zero-monotonic games are both zero-monotonic games as well.

The Shapley value on size zero-monotonic M-games

Since we know that the worths of all the singleton coalitions are zero, the definition of a size zero-monotonic N-game ω can be stated as $\omega(S) \le \omega(T)$ for all $S, T \subseteq N$ satisfying $|S| < |T|$. This indicates that the M-game is size zero-monotonic if every 2-player coalition is less than or equal to every 3-player coalition which is less than or equal to every 4-player coalition, and so on, up to the grand coalition. The set of size zero-monotonic extensions of an M-game is defined by $\mathrm{ext}(\omega) = \{\hat{\omega}:\hat{\omega}(S)=\omega(S)$ if $|S|\in M$, $\max\{\hat{\omega}(T):|T|=|S|-1\} \leq \hat{\omega}(S) \leq \min\{\hat{\omega}(T):|T|=|S|+1\}$ if $|S|\notin M\}$.

To determine the centroid of $\mathrm{ext}(\omega)$, look at the $\hat{\omega}(S)$ when $|S|\notin M$. The value assigned to $\hat{\omega}(S)$ depends solely on the cardinality of S, with no means of differentiation based on the composition of the coalition. In other words, the range of $\hat{\omega}(S)$ for each S with the same number of players (regardless of which players are in the coalition) looks exactly the same. Looking at a cross section of the object created when $|S|=s$, with all other $\hat{\omega}(S)$ fixed, a box is formed in multidimensional space. The centroid of this object must then have the same value for each $\omega(S)$ within a cardinality. So, the $\mathbf{C}(S)$ is the same within each cardinality where $|S| \notin M$.

Taking the Shapley value of the centroid of the size zero-monotonic extension of the M-game, denoted by θ, provides us with an easily generalized allocation method. In this extension, the Shapley value of the centroid extension is equivalent to taking the Shapley value of the partially defined game with respect to M by simply ignoring the worths of the coalitions whose size is not in M, since

$$\theta_i(\mathbf{C}) = \varphi_i(\omega) = \frac{1}{n}\sum_{|S|\in N} \left[\sum_{|S|=s,\, i\in S}\binom{n-1}{s-1}^{-1}\mathbf{C}(S) - \sum_{|S|=s,\, i\notin S}\binom{n-1}{s}^{-1}\mathbf{C}(S)\right]$$

This is obtained from the formula stated earlier for N-games in which the first summation was over $|S| \in N$, rather than M. Looking at the term in brackets for $|S|=s \notin M$, we know that the $\mathbf{C}(S)$ is the same whether or not i is in S. Since the first sum has $\binom{n-1}{s-1}$ terms and the second sum has $\binom{n-1}{s}$ terms, the expression in brackets reduces to

$$\binom{n-1}{s-1}\binom{n-1}{s-1}^{-1}\mathbf{C}(S)-\binom{n-1}{s}\binom{n-1}{s}^{-1}\mathbf{C}(S)=0.$$

So, when $|S| \notin M$, the term in brackets will always equal zero and can, therefore, be ignored. When $|S| \in M$, the worth of the centroid extension is equal to the worth of the original M-game ($\mathbf{C}(S) = \omega(S)$). So the allocation for a size zero-monotonic M-game is:

$$\varphi_i(\omega)=\frac{1}{n}\sum_{|S|\in M}[\sum_{|S|=s,i\in S}\binom{n-1}{s-1}^{-1}\omega(S)-\sum_{|S|=s,i\notin S}\binom{n-1}{s}^{-1}\omega(S)]$$

In a different notational form, Wilson refers to the above formula as the reduced Shapley value [Wilson, 1991].

The Shapley value on zero-monotonic M-games

Since we assume that the M-game ω is already zero-normalized, the definition of a zero-monotonic game ω can be simplified to $\omega(S) \le \omega(T)$ for all $S \subseteq T \subseteq N$. Like the size zero-monotonic extension, this insures that adding a player to a coalition does not lessen the worth of the original coalition. But, unlike the size zero-monotonic extension, the zero-monotonic extension allows for the possibility that a 3 player coalition could be less than some 2 player coalition. The set of zero-monotonic extensions is defined by ext(ω) = $\{\hat{\omega}:\hat{\omega}(S)=\omega(S) \text{ if } |S|\in M, \max\{\hat{\omega}(T): T\subseteq S\} \le \hat{\omega}(S) \le \min\{\hat{\omega}(T): T\supseteq S\} \text{ if } |S|\notin M\}$. Finding the centroid extension on this class of games is not a simple matter. In cases where two or more successive coalitional sizes are absent from M, the $\hat{\omega}(S)$ can be difficult to determine and even harder to generalize for the n case. I was able to find results for some of the simpler cases. For 4-player zero-monotonic games where M = {1,2,4}, the shapley value of the centroid ext(ω) is

$$\theta_i(C) = \varphi_i(\omega) = \frac{1}{n}\sum_{|S|\in M} [\Sigma_{|S|=s,\, i\in S}\binom{n-1}{s-1}^{-1}\omega(S) - \Sigma_{|S|=s,\, i\notin S}\binom{n-1}{s}^{-1}\omega(S)]$$

$$+\frac{1}{n}\sum_{|S|\in M}\frac{1}{2}[\Sigma_{|S|=s,\, i\in S}\binom{n-1}{s-1}^{-1}\max\{\omega(T): T\subseteq S\} - \Sigma_{|S|=s,\, i\notin S}\binom{n-1}{s}^{-1}\max\{\omega(T): T\subseteq S\}]$$

For 4-player zero-monotonic games where M = {1,3,4}, the shapley value of the centroid ext(ω) is

$$\theta_i(C) = \varphi_i(\omega) = \frac{1}{n}\sum_{|S|\in M} [\Sigma_{|S|=s,\, i\in S}\binom{n-1}{s-1}^{-1}\omega(S) - \Sigma_{|S|=s,\, i\notin S}\binom{n-1}{s}^{-1}\omega(S)]$$

$$+\frac{1}{n}\sum_{|S|\in M}\frac{1}{2}[\Sigma_{|S|=s,\, i\in S}\binom{n-1}{s-1}^{-1}\min\{\omega(T): T\supseteq S\} - \Sigma_{|S|=s,\, i\notin S}\binom{n-1}{s}^{-1}\min\{\omega(T): T\supseteq S\}]$$

If there were some generalized means of finding the centroid of the object defined by ext(ω), then possibly the shapley value could be generalized to some simple formula, but I was unable to discover such a method.

The Shapley value on superadditive M-games

The superadditive extensions of the M-game ω, where $\omega(S\cup T) \geq \omega(S) + \omega(T)$ for all disjoint $S,T \subseteq N$, is difficult to characterize. In the 4-player game where M={1,3,4}, the set of superadditive extensions look like the zero-monotonic extensions with the added constraints: $\omega(12) + \omega(34) \leq \omega(1234)$, $\omega(13) + \omega(24) \leq \omega(1234)$, and $\omega(14) + \omega(23) \leq \omega(1234)$. To facilitate notation, when looking at the known coalitional worths for $|S| = 3$, order the worths so that $\omega(123) \geq \omega(124) \geq \omega(134) \geq \omega(234)$. In superadditive games, the extensions take on different shapes (and therefore, different centroid values) depending on the worths of the known coalitions. In the first case, if $\omega(1234) \geq \omega(124) + \omega(234)$, the superadditive extensions become equivalent to the zero-monotonic extensions. The centroid ext(ω) and the corresponding shapley value are found as in the previous section. In the second and third cases the $\omega(1234) < \omega(124) + \omega(234)$ and either $\omega(1234) \geq \omega(134) + \omega(234)$ or $\omega(1234) < \omega(134) + \omega(234)$. In both cases the centroid does not generalize easily. While I did find the centroid ext(ω) of this particular game, I was not able to see a generalized formula for either the centroid ext(ω) or the shapley value of the centroid ext(ω) [Appendix A,B].

Summary of centroid ext(ω)

For any particular M-game ω, the class of games chosen to define the extension $\hat{\omega}$ determines the resulting allocation, $\varphi_i(\hat{\omega})$. To illustrate the differences of the preceding results, below is a table comparing the size zero-monotonic, zero-monotonic, and superadditive extensions and their shapley values for the game ω where: $\omega(1234) = 240$; $\omega(123) = 200$; $\omega(124) = 180$; $\omega(134) = 160$; $\omega(234) = 120$; and $\omega(i) = 0$ for $i \in \{1,2,3,4\}$.

	size zero-monotonic	zero-monotonic	superadditive
$\hat{\omega}(12)$	60	90	83.6364
$\hat{\omega}(13)$	60	80	77.1014
$\hat{\omega}(14)$	60	80	77.1014
$\hat{\omega}(23)$	60	60	57.9710
$\hat{\omega}(24)$	60	60	57.9710
$\hat{\omega}(34)$	60	60	56.3636
$\varphi_1(\hat{\omega})$	75	80.83	80.4611
$\varphi_2(\hat{\omega})$	61.67	60.83	60.7510
$\varphi_3(\hat{\omega})$	55	52.5	52.7273
$\varphi_4(\hat{\omega})$	48.33	45.83	46.0606

An Alternative to the centroid extension

Since the centroid appears to be a difficult extension to find in the zero-monotonic class of games as n gets large, let's look at a simpler central point. Define x such that x is a *coordinate center* of the convex set C if x_i

is the midpoint of $\{x + \lambda\hat{e}_i : \lambda \in \mathbb{R}\} \cap C$ for all i. It can be shown that in the zero-monotonic extension of the M-game ω, there exists a **unique** coordinate center [Housman, appendix C]. Let $\hat{c}$ be the coordinate center of ext(ω), then $\hat{c} = \{\hat{c}: \hat{c}(S) = \omega(S)$ if $|S| \in M$, $\hat{c}(S) = (1/2)[\max\{\hat{c}(R) : R \subset S\} + \min\{\hat{c}(T) : S \subset T\}]$ if $|S| \notin M\}$.

At this point it becomes useful to narrow the type of game with which we are working. Oftentimes, the worths of the coalitions other than the grand coalition and the coalition minus one member are not available. Considering M-games of this nature, where M ={1, n-1, n}, has many real-life applications in the cost allocation of joint ventures. So, an M-game ω where M = {1, n-1, n}, can be ordered for notational purposes so that $\omega(N - \{i\}) = a_i$ and $a_1 \leq a_2 \leq \ldots \leq a_n \leq a_0 = \omega(N)$. In this case, the coordinate center, $\hat{c}$, can be simplified to: $\hat{c} = \{\hat{c}: \hat{c}(S) = \omega(S)$ if $|S| \in M$,

$\hat{c}(S) = \frac{s-1}{n-2}\min\{a_i : i \notin S\}$ if $S \neq \{1,2,\ldots,s\}$, $|S| \notin M$, $\hat{c}(S) =$

$\frac{1}{n-2}\sum \frac{k}{2^{s-k}} a_{k+2}$ if $S = \{1,2,\ldots,s\}$, $|S| \notin M$ }. In this form, generalizing the shapley value on this game is likely to simplify. If so, this would provide a much more efficient means of calculating an allocation using the shapley value then determining all possible coalitional worths first. Due to a lack of time, I am unable to say for certain if this is the case.

Conclusion

Although many of the results submitted in this paper are only preliminar y findings which may not yet be fully developed, I believe that there is merit in the procedure put forth. Defining a set of extensions based on a class of games, using this to determine a central extension, and then applying the Shapley formula to these values will, at least in some specific cases, lead to a simple allocation formula based on only the known coalitional worths of the partially defined game with respect to M. The allocation determined in the size zero-monotonic class of M-games is the most conclusive, and may be applicable in many reality based problems, but it must be noted that it also imposes some severe restrictions on the game.

References

Housman, David. Mathematics Department, Alleghany College. Verbal consultation.

Ventrudo, Tom, and Wallman, Jodi. "Finding a Value on Partially Defined Games," August, 1991.

Wilson, Stephen J. "A Value for Partially Defined Cooperative Games," International Journal of Game Theory, Vol 21, Issue 4. 1993 Physica - Verlag, Heidelberg.

APPENDIX A

Superadditive M = {1,3,4}

$n=4 \quad M = \{1,3,4\}$

S	w(S)
1234	a
123	b
124	c
134	d
234	e

$a \geq b \geq c \geq d \geq e$

w(12)

`xx1;`

$$1/3\,(3a^2e - 3a^2\min(a-c,e) - 3ae^2 + 3a\min(a-c,e)^2 + e^3 - \min(a-c,e)^3 + 3c^2\min(a-c,e)) \Big/ (2ea - 2a\min(a-c,e) - e^2 + \min(a-c,e)^2 + 2c\min(a-c,e))$$

w(13) = w(14)

`xx2;`

$$1/3\,(3a^2e - 3a^2\min(a-d,e) - 3ae^2 + 3a\min(a-d,e)^2 + e^3 - \min(a-d,e)^3 + 3d^2\min(a-d,e)) \Big/ (2d\min(a-d,e) + 2ea - e^2 - 2\min(a-d,e)a + \min(a-d,e)^2)$$

=

`xx3;` w(13) = w(14)

$$1/3\,(3a^2e - 3a^2\min(a-d,e) - 3ae^2 + 3a\min(a-d,e)^2 + e^3 - \min(a-d,e)^3 + 3d^2\min(a-d,e)) \Big/ (2d\min(a-d,e) + 2ea - e^2 - 2\min(a-d,e)a + \min(a-d,e)^2)$$

`xx4;` w(23) = w(24)

$$1/3\,\frac{3d\min(a-d,e)^2 - 2e^3 + 3ae^2 + 2\min(a-d,e)^3 - 3a\min(a-d,e)^2}{2d\min(a-d,e) + 2ea - e^2 - 2\min(a-d,e)a + \min(a-d,e)^2}$$

=

`xx5;` w(24) = w(23)

$$1/3\,\frac{3d\min(a-d,e)^2 - 2e^3 + 3ae^2 + 2\min(a-d,e)^3 - 3a\min(a-d,e)^2}{2d\min(a-d,e) + 2ea - e^2 - 2\min(a-d,e)a + \min(a-d,e)^2}$$

`> xx6;` w(34)

$$1/3\,\frac{3ae^2 - 3a\min(a-c,e)^2 - 2e^3 + 3c\min(a-c,e)^2 + 2\min(a-c,e)^3}{2ea - 2a\min(a-c,e) - e^2 + \min(a-c,e)^2 + 2c\min(a-c,e)}$$

```
> ;
> ;
> ;
> simplffy(subs(min(a-c,e)=e,min(a-d,e)=e,xx1);
syntax error:
simplify(subs(min(a-c,e)=e,min(a-d,e)=e,xx1);
                                             ^
> simplify(subs(min(a-c,e)=e,min(a-d,e)=e,xx1));
                                     1/2 c

> simplify(subs(min(a-c,e)=e,min(a-d,e)=e,xx2));
                                     1/2 d
```

Case I: $a \geq c+e$
$a \geq d+e$

$w(12) = \frac{1}{2}c$

$w(13) = w(14) = \frac{1}{2}d$

$w(23) = w(24) = w(34) = \frac{1}{2}e$

Case II: $a < c+e$, $a < d+e$

```
sx1;
```

$$\frac{1}{3}\,\frac{-3a^2e+a^3+3ae^2-3ac^2-e^3+2c^3}{-2ea+a^2-2ac+e^2+c^2} = w(12)$$

```
> sx2;
```

xx2
xx3

$$\frac{1}{3}\,\frac{-3a^2e+a^3+3ae^2-3ad^2-e^3+2d^3}{-2ad+d^2-2ea+e^2+a^2} = w(13) = w(14)$$

```
> sx4;
```

xx4
xx5

$$\frac{1}{3}\,\frac{-3a^2d+3ad^2-d^3+2e^3-3ae^2+a^3}{-2ad+d^2-2ea+e^2+a^2} = w(23) = w(24)$$

```
> sx6;
```

$$\frac{1}{3}\,\frac{-3ae^2+a^3-3a^2c+3ac^2+2e^3-c^3}{-2ea+a^2-2ac+e^2+c^2} = w(34)$$

```
> sx8;= simplify(subs(min(a-c,e)=(a-c),min(a-d,e)=(a-d),xx6));
>
```

Case III: $a < c+e$, $a \geq d+e$

$$w(12) = \frac{1}{3}\,\frac{-3a^2e+a^3+3ae^2-3ac^2-e^3+2c^3}{-2ea+a^2-2ac+e^2+c^2}$$

$$w(13) = w(14) = \frac{1}{2}d$$

$$w(23) = w(24) = \frac{1}{2}e$$

$$w(34) = \frac{1}{3}\,\frac{-3ae^2+a^3-3a^2c+3ac^2+2e^3-c^3}{-2ea+a^2-2ac+e^2+c^2}$$

APPENDIX B

```
shapley values;
syntax error:
shapley values;
               ^
MORE> ;
> sh1;
```

Case II : Shapley Values

$$\frac{1}{12}\big(-7ac^2d^2+5ae^2d^2+9ac^2e^2+6be^2a^2-2ba^3c+3a^5-6e^5$$
$$+12ea^3d+2ea^2d^2+2a^3cd+8a^2cd^2+16a^3ce-19a^2ce^2$$
$$-14e^2a^2d+5c^2a^2d-3c^2ea^2+6ace^3+4e^3ad-3ec^2d^2$$
$$-6ead^3-6acd^3-4ea^2cd+4eacd^2+4ec^2ad+27ae^4$$
$$+42e^2a^3-48e^3a^2-18ea^4-3ad^4-3a^3d^2-4ac^4-c^2a^3$$
$$-4e^3d^2-5c^2e^3+de^4+3a^2d^3+3e^2d^3+3c^2d^3+2c^3d^2$$
$$+2c^3e^2+2c^3a^2+ce^4+ba^4+be^4-4dace^2+dc^2e^2-4c^3ad$$
$$-4c^3ea+ce^2d^2-4bea^3-4bea^3-2ba^3d+ba^2d^2+be^2d^2$$
$$+4bea^2d-2bead^2+4ba^2cd-2bacd^2+4ba^2ce$$
$$-2bace^2-2be^2ad-2bc^2ad-2bc^2ea+bc^2d^2+bc^2e^2$$
$$+bc^2a^2\big)$$
$$\Big/\ \big((-2ea+a^2-2ac+e^2+c^2)(-2ad+d^2-2ea+e^2+a^2)\big)$$

$(\mathcal{U}_1(\omega))$

```
> sh2;
```

$$\frac{1}{12}\big(9ac^2d^2+13ae^2d^2-7ac^2e^2+6be^2a^2-2ba^3c+3a^5+2e^5$$
$$+28ea^3d-26ea^2d^2+18a^3cd-24a^2cd^2+13a^2ce^2$$
$$-26e^2a^2d-3c^2a^2d+5c^2ea^2-10ace^3+12e^3ad+ec^2d^2$$
$$+10ead^3+10acd^3-4ea^2cd-4eacd^2+4ec^2ad-5ae^4$$
$$+10e^2a^3-10ea^4-11ad^4+13a^3d^2-4ac^4-c^2a^3+3c^2e^3$$
$$-3de^4-5a^2d^3-5e^2d^3-5c^2d^3+2c^3d^2+2c^3e^2+2c^3a^2$$
$$+ce^4+ba^4+be^4+4dace^2-3dc^2e^2-4c^3ad-4c^3ea,$$

2 2 3 3 3 2 2 2 2

$(\mathcal{U}_2(\omega))$

$+ c e^2 d^2 - 4 b e^3 a - 4 b e a^3 - 2 b a^3 d + b a^2 d^2 + b e^2 d^2$

$+ 4 b e a^2 d - 2 b e a d^2 + 4 b a^2 c d - 2 b a c d^2 + 4 b a^2 c e$

$- 2 b a c e^2 - 2 b e^2 a d - 2 b c^2 a d - 2 b c^2 e a + b c^2 d^2 + b c^2 e^2$

$+ b c^2 a^2)$

$/ ((- 2 a d + d^2 - 2 e a + e^2 + a^2)(- 2 e a + a^2 - 2 a c + e^2 + c^2))$

> sh3; → $\theta_C(\omega)$ $\qquad (\alpha_3(\omega))$

$1/12\,(- 2 b e a + 4 c e a - 4 a^2 e - a e^2 + 2 e^3 - 4 c^3 + c^2 e + 3 a^3 - 10 a^2 c$

$+ 11 a c^2 + a^2 d + b e^2 + b c^2 - 3 c^2 e + e^2 d + c^2 d + b a^2 - 2 b a c$

$- 2 d e a - 2 d a c) \;/\; (- 2 e a + a^2 - 2 a c + e^2 + c^2)$

> sh4; → $\theta_D(\omega)$ $\qquad (\alpha_4(\omega))$

$1/12\,(6 b e a - 4 c e a - 4 a^2 e - a e^2 + 2 e^3 + c^2 e + 3 a^3 - 6 a^2 c + 3 a c^2$

$+ a^2 d - 3 b e^2 - 3 b c^2 + c e^2 + e^2 d + c^2 d - 3 b a^2 + 6 b a c - 2 d e a$

$- 2 d a c) \;/\; (- 2 e a + a^2 - 2 a c + e^2 + c^2)$

>

APPENDIX B (cont.)

Case III
Shapley Values

```
> sx6:=simplify(subs(min(a-c,e)=(a-c),min(a-d,e)=e,xx6)):
> sh1;
```

$\varphi_1(\omega)$

$$1/12\,(-2bea+6cea-11a^2e+13ae^2-5e^3+2c^3-4c^2e+3a^3-4a^2c-ac^2+2a^2d+be^2+bc^2+ce^2+2e^2d+2c^2d+ba^2-2bac-4dea-4dac)\,/\,(-2ea+a^2-2ac+e^2+c^2)$$

```
> sh2;
```

$\varphi_2(\omega)$

$$1/12\,(-2bea-6cea-5a^2e+ae^2+e^3+2c^3+2c^2e+3a^3-4a^2c-ac^2-4a^2d+be^2+bc^2+ce^2-4e^2d-4c^2d+ba^2-2bac+8dea+8dac)\,/\,(-2ea+a^2-2ac+e^2+c^2)$$

```
> sh3;
```

$\varphi_3(\omega)$

$$1/12\,(-2bea+4cea-4a^2e-ae^2+2e^3-4c^3+c^2e+3a^3-10a^2c+11ac^2+a^2d+be^2+bc^2-3ce^2+e^2d+c^2d+ba^2-2bac-2dea-2dac)\,/\,(-2ea+a^2-2ac+e^2+c^2)$$

```
> sh4;
```

$\varphi_4(\omega)$

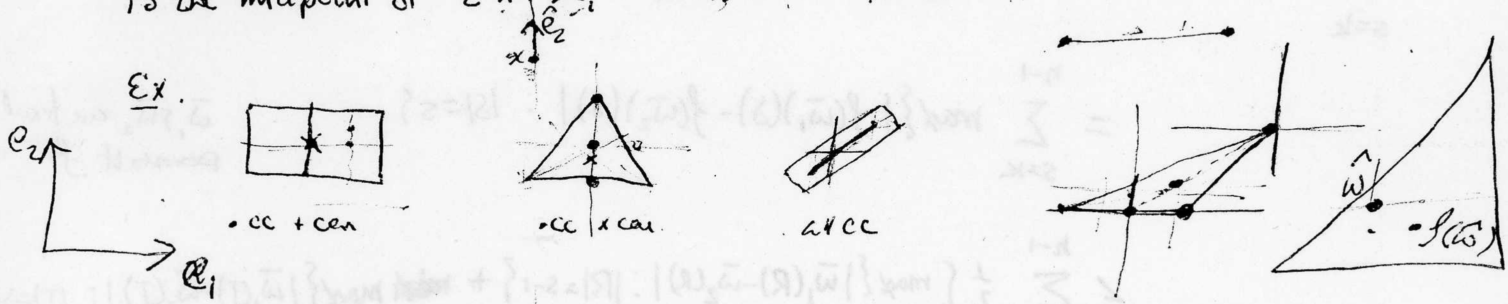
$$1/12\,(6bea-4cea-4a^2e-ae^2+2e^3+c^2e+3a^3-6a^2c+3ac^2+a^2d-3be^2-3bc^2+ce^2+e^2d+c^2d-3ba^2+6bac-2dea-2dac)\,/\,(-2ea+a^2-2ac+e^2+c^2)$$

```
>
```

APPENDIX C*

Def'n. x is a coordinate center of the convex set C if x_i is the midpoint of $\{x + \lambda \hat{e}_i : \lambda \in \mathbb{R}\} \cap C$ for all i.

Ex.



Necess. & suff. condition so that

Consider the zeromonotonic ~~[illegible]~~ M-game ω. ~~Let~~ $\bar{\omega}$ ~~be~~ is a coorcent. of $Ext(\omega)$. ~~That $\bar{\omega}$ must satisfy~~ are

$$\bar{\omega}(S) = \frac{1}{2}\left[\max\{\bar{\omega}(R) : R \subset S\} + \min\{\bar{\omega}(T) : S \subset T\}\right] \quad \forall |S| \notin M$$

$$\bar{\omega}(S) = \omega(S) \quad \forall |S| \in M$$

Define the map f on $Ext(\omega)$ by

$$f(\hat{\omega})(S) = \begin{cases} \frac{1}{2}\left[\max\{\hat{\omega}(R) : R \subset S\} + \min\{\hat{\omega}(T) : S \subset T\}\right] & \text{if } |S| \notin M \\ \hat{\omega}(S) & \text{if } |S| \in M \end{cases}$$

Note that $f(\hat{\omega}) \in Ext(\omega)$ because if $S_1 \subseteq S_2$ $(|S_1|, |S_2| \notin M)$ then $f(\hat{\omega})(S_1) \leq \frac{1}{2}\left[\hat{\omega}(S_1) + \hat{\omega}(S_2)\right] \leq f(\hat{\omega})(S_2)$. Clearly, the fixed points of f are the coordinate centers of $Ext(\omega)$. It is also clear that f is continuous (in fact, Lipschitz / nonexpansive map: $\|f(\hat{\omega}_1) - f(\hat{\omega}_2)\|_\infty \leq \|\hat{\omega}_1 - \hat{\omega}_2\|_\infty$) and so Brouwer's Fixed Point Theorem implies that f has fixed points. Suppose $\bar{\omega}_1$ & $\bar{\omega}_2$ are both fixed points. Of course, $\bar{\omega}_1(S) = \bar{\omega}_2(S)$ for $|S| = 1$. Now suppose $\bar{\omega}_1(R) = \bar{\omega}_2(R)$ for all $|R| \leq$ ~~s~~ $k-1$. Then for $|S| = s \notin M$,

~~$|\bar{\omega}_1(S) - \bar{\omega}_2(S)| = |f(\bar{\omega}_1)(S) - f(\bar{\omega}_2)(S)| = \left|\frac{1}{2}\left[\min\{\bar{\omega}_1(T) : S \subset T\} - \min\{\bar{\omega}_2(T) : S \subset T\}\right]\right|$~~

$$\leq \frac{1}{2}\|\bar{\omega}_1 - \bar{\omega}_2\|_\infty$$

*provided by David Housman, Allegheny College

$$\sum_{s=k}^{n-1} \max\{|\bar{w}_1(S)-\bar{w}_2(S)| : |S|=s\}$$

$$= \sum_{s=k}^{n-1} \max\{|f(\bar{w}_1)(S) - f(\bar{w}_2)(S)| : |S|=s\}$$ $\bar{w}_1$ & $\bar{w}_2$ are fixed points of f

$$\leq \sum_{s=k}^{n-1} \tfrac{1}{2}\left[\max\{|\bar{w}_1(R)-\bar{w}_2(R)| : |R|=s-1\} + \max\{|\bar{w}_1(T)-\bar{w}_2(T)| : |T|=s+1\}\right]$$

$$\leq \tfrac{1}{2}\max\{|\bar{w}_1(S)-\bar{w}_2(S)| : |S|=k\} + \sum_{s=k+1}^{n-1} \max\{|\bar{w}_1(S)-\bar{w}_2(S)| : |S|=s\}$$

$\Rightarrow$ $\max\{|\bar{w}_1(S)-\bar{w}_2(S)|, |S|=k\} = 0$ $\Rightarrow$ $\bar{w}_1(S)=\bar{w}_2(S)$ for $|S|=k$.

By induction $\bar{w}_1 = \bar{w}_2$. Thus, there is only one c.c. for $Ext(W)$.

DOES THIS GENERALIZE TO OTHER CLASSES OF GAMES?

Find $\max\{\varphi_i(\hat{w}) : \hat{w} \in Ext(W)\}$ ← Could "average" these "vertices"
min

Steiner point (Grünbaum)