Monotonic Power Indices

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Abstract

United States legislation requires approval by the President and simple majorities of the Representatives and of the Senators, or approval by two-thirds majorities of the Representatives and of the Senators. Four well-known power indices assign the President between 4% and 77% of the a priori voting power in this simple game. Given this wide range of answers and a variety of power index paradoxes cited in the literature, it is important to clearly define what properties a power index should satisfy. The goal of this work is to characterize all power indices satisfying the normalization, symmetric, dummy independence, and monotonicity properties.

1. Introduction

In the United States government, legislation either requires approval by the President and simple majorities of the Representatives and of the Senators or approval by two-thirds majorities of the Representatives and of the Senators. This is an example of a simple voting game: there are voters and winning coalitions (groups of voters that can pass a resolution). There are three characteristics that every simple voting game must have. First, all voters together must form a winning coalition. Second, if a subset of a group of voters is a wining coalition, then the group of voters is also a winning coalition. For example, in a game consisting of 4 players if voters A and B form a winning coalition, then voters A, B, and C must also form a winning coalition. Lastly, in simple voting games, there can't be two disjoint coalitions that could win at the same time. For example, in a game consisting of 4 players if voters A and B form a winning coalition, then voters C and D can't form a winning coalition because in this case, they would both be able to win at the same time. It doesn't make sense in real life so it is not allowed to happen.

There are several different types of games. A dictatorship is when one player has all the power because he forms a winning coalition by himself. In this game, no other players need to approve for the resolution to be passed. Another game is when all the players are part of the same number of winning coalitions, so they have equal power. Then there are the games that are in between these two games.

From these simple voting games, the voting power of each voter can be calculated. Different methods of calculating voting power of each voter have been created. These different methods are called power indices. The four most popular power indices are the Banzhaf, Deegan-Packel, Shapley-Shubik, and Johnston power indices. When looking at United States legislation, the President is assigned 4%, 4%, 16%, and 77% of the power respectively. This is a large range of power ascribed to the President. Arguments exist about which of these power indices is the most correct and the best for calculating voting power in simple voting games. Voting power can be considered the likelihood of having a significant role in determing the outcome of a vote.

Certain reasonable properties exist that can be used to help argue which power index is the best for calculating voting power in simple voting games [2]. First, the voting powers of all the players should add up to one. The power index can then be interpreted as the fraction of voting power that a voter has, which helps when comparing a voter's power between different games. Next if the voters in a game are relabeled then the powers of those voters should also be relabeled. This means that if voter A is relabeled as voter B, voter B is relabeled as voter C, and voter C is relabeled as voter A, then the new voter A has the amount of voting power that voter C had originally, the new voter B has the amount of voting power that voter A had originally, and the new voter C has the voting power that voter B originally had. Third, if a player can never change a winning coalition to a losing coalition by removing himself from the coalition or change a losing coalition to a winning coalition by adding himself to a coalition, then he does not have any voting power at all. Since that player can never change any of the winning coalitions, then essentially his vote doesn't matter.

A fourth reasonable property shows up when two games are compared. If the only difference between two games is that the first game has a winning coalition that is not present in the second game, then the voters in the additional winning coalition in the first game should have more power in the first game than they do in the second game. Not all of the aforementioned power indices follow all of these properties. In fact only the Shapley-Shubik index follows all the properties mentioned. The other three power indices violate the last property. The goal of this study is to find all the power indices that follow these properties. This would be a first step to finding the best way to calculate voting power in simple voting games.

2. The Problem

The problem addressed in this paper is how to characterize all power indices, for simple voting games, that follow the properties of normalization, symmetry, dummy, dummy independence, and monotonicity. For our purposes, voting power of a player is a measurement of that player having a significant role in determining the outcome of a vote. First, the definition of simple voting games used in this study must be defined.

Definition 1 *A simple voting game is a set of players N and a set of winning coalitions W satisfying*

1. $N \in W$,
2. $\emptyset \notin W$,
3. *if* $S \in W$ *and* $S \subset T$ *then* $T \in W$, *and*
4. *if* $S \in W$ *and* $T \in W$ *then* $S \cap T \neq \emptyset$.

The following example will help to clarify what a simple voting game is. $M = [t; p_1, p_2, p_3, ..., p_n]$ is the notation used in a weighted voting game. Weighted voting games are a subset of simple voting games. In weighted voting games, each voter is assigned a weight for which their vote is worth. For example, if player A is assigned a weight of 5, her vote counts as 5 votes instead of 1. These assigned weights may be because each voter is representing a different number of people. In the notation given, the game M has n players (or voters) with weights of $p_1, p_2, p_3, ...,$ and p_n. A minimum of t votes is needed for a group of voters to form a winning coalition.

Example 2 $M = [8; 5, 3, 1, 1, 1]$

This game has 5 voters whom I will call A, B, C, D, and E. Because at least 8 votes are needed to form a winning coalition, the only possible winning coalitions are

$$ABCDE \quad ABCD \quad ABCE \quad ABDE \quad ABC \quad ABD \quad ABE \quad AB \quad ACDE$$

From these winning coalitions, we want to find the voting power that each voter has. There are different methods of calculating this voting power which are called voting power indices. A power index assigns a power to each voter.

Definition 3 *A power index is a function that associates with each simple voting game (N, W) and voter $i \in N$ a number $K_i(N, W)$.*

The two power indices that I looked at during my research were the Banzhaf and the Shapley-Shubik which are named after the persons who created them. Both of these power indices look at which players are critical. A player is critical when he is able to change a winning coalition to a losing coalition by removing himself from the coalition or when he is able to change a losing coalition to a winning coalition by adding himself to the coalition. The Banzhaf power index looks at all the critical voters in each winning coalition, adds up the number of winning coalitions that each voter is critical in, and then divides that number by the total for all the voters. The critical voters in the above example are underlined below:

$$\underline{\text{A}}\text{BCDE} \quad \underline{\text{AB}}\text{CD} \quad \underline{\text{AB}}\text{CE} \quad \underline{\text{AB}}\text{DE} \quad \underline{\text{ABC}} \quad \underline{\text{ABD}} \quad \underline{\text{ABE}} \quad \underline{\text{AB}} \quad \underline{\text{ACDE}}$$

Voter A is critical in all the winning coalitions so she has a voting power of $\frac{9}{19}$. Voter B is critical in 7 out of the 9 winning coalitions so he has a voting power of $\frac{7}{19}$. Voter C,D,and E all were critical in only 1 winning coalition so they each have a voting power of $\frac{1}{19}$.

The Shapley-Shubik power index looks at all the different orders that the voters can be arranged in and finds the one player that makes the coalition a winning coalition in each of the different orderings. Then the number of times a player is critical is added up and divided by the total number orderings to calculate the voting power of each voter. In the above example, there are a total of 120 possible orderings of the 5 voters. Voter A is critical in 66 of the 120 orderings so she has a voting power of $\frac{66}{120}$, voter B is critical in 36 of the 120 orderings so he has a voting power of $\frac{36}{120}$, and voters C, D, and E were each critical in 6 of the 120 orderings so they each have a voting power of $\frac{6}{120}$. By looking at these powers calculated from both the Banzhaf power index and the Shapley-Shubik power index, you can see that different power indices can produce different values for the voting power of each voter. This can raise the question of which power index is the best for calculating voting power.

There are certain properties that make logical sense that any power index should follow. These properties are normalization, symmetry, dummy, dummy independence, and monotonicity. The first property gives a relative voting power for each voter by making all the individual powers add up to 1. This is so it can be used when making comparisons both with the other voters in the same game and with the same voter in different games.

Definition 4 (Normalization) *If (N, W) is a simple voting game, then* $\sum_{i=1}^{n} K_i\,(N, W) = 1$.

Renaming the voters in a simple voting game should not change their powers. We can formalize renaming with a permutation. A *permutation* π is a one-to-one mapping of N onto N. For example, let $N = \{A, B, C, D, E\}$ and $\pi(A) = B$, $\pi(B) = C$, $\pi(C) = D$, $\pi(D) = E$, and $\pi(E) = A$. Let (N, W) be any simple voting game and π a permutation of N. We denote the corresponding renaming of a coalition S by $\pi(S) = \{\pi(i) : i \in S\}$. We denote the corresponding renaming of the winning coalitions W by $\pi(W) = \{\pi(S) : S \in W\}$. Using the example above with the simple voting game M and the permutations specified previously, $\pi(W) = \{\{ABCDE\}, \{BCDE\}, \{BCDA\}, \{BCEA\}, \{BCD\}, \{BCE\}, \{BCA\}, \{BC\}, \{BDEA\}\}$. The second property renames the voting power of each voter when the voters are renamed.

Definition 5 (Symmetry) *If (N, W) is a simple voting game and π is a permutation of N, then $K_i(W) = K_{\pi(i)}(\pi(W))$ for every voter $i \in N$.*

A voter who can never change a losing coalition to a winning coalition is considered a dummy. That is, voter $i \in N$ is a *dummy* in the simple voting game (N, W) if $S \in W$ if and only if $S \setminus \{i\} \in W$ for all coalitions $S \subseteq N$.

Dummy voters play a special role in simple voting games. First, since dummy voters can't affect the outcome of any vote, then they should not have any voting power.

Definition 6 (Dummy) *If (N, W) is a simple voting game and voter i is a dummy, then $K_i(N, W) = 0$.*

Second, because a dummy does not affect the outcome of any vote, the same amount of voting power should be assigned to the other voters if the dummy is excluded from the simple voting game.

Definition 7 (Dummy Independence) *If (N', W') is a simple voting game obtained by excluding a dummy from a simple voting game (N, W), then $K_i(N', W') = K_i(N, W)$ for every voter $i \in N'$.*

The last property increases the voting power of voters when they are part of more winning coalitions and decreased the voting power of voters when they are part of less winning coalitions.

Definition 8 (Monotonicity) *If (N, W) and (N', W') are simple voting games such that $W' = W \cup \{S\}$ then $K_i(N', W') \geq K_i(N, W)$ for every voter $i \in S$ and $K_i(N', W') \leq K_i(N, W)$ for every voter $i \in N \setminus S$.*

All of these properties are reasonable in any simple voting game. The Banzhaf power index and the Shapley-Shubik power index both follow the first four properties: normalization, symmetry, dummy, and dummy independence. This can be illustrated for the first two properties using the above example. The sum of the voting powers of each voter add up to 1 in the first example using each power index. Thus, both the Banzhaf and Shapley-Shubik power indices satisfy normalization. Also in the first example, renaming voters C, D, and E doesn't change any of the winning coalitions, therefore the voting powers of these voters must be the same. This is also seen when voting power is calculated by both the Banzhaf and Shapley-Shubik power indices as shown above so they also satisfy symmetry. Because the above example does not illustrate the dummy and dummy independence properties, here is an example that exhibits these.

Example 9 $N = [8; 4, 4, 1, 1, 1]$

This game is similar to the first example except that voter A's vote now counts as 4 votes instead of 5 and voter B's vote now counts as 4 votes instead of 3. From the information given, we know that these are the only possible winning coalitions (with the critical voters underlined):

$$\underline{A}\underline{B}CDE \quad \underline{A}\underline{B}CD \quad \underline{A}\underline{B}CE \quad \underline{A}\underline{B}DE \quad \underline{A}\underline{B}C \quad \underline{A}\underline{B}D \quad \underline{A}\underline{B}E \quad \underline{A}\underline{B}$$

Using the Banzhaf power index to calculate voting power, voters A and B each have a voting power of $\frac{1}{2}$ while voters C, D, and E each have a voting power of 0. Using the Shapley-Shubik power index to calculate voting power, voters A and B each have a voting power of $\frac{60}{120}$ or $\frac{1}{2}$ while voters C, D, and E each have a voting power of 0. By looking at the winning coalitions, we see that C, D, and E are dummy player, so they must all have a voting power of 0. Both the Banzhaf and Shapley-Shubik power indices calculate their power to be 0 so they satisfy the dummy property.

The last property to consider is the monotonicity property. This property is shown when comparing the two examples above. The difference in those two games is that game M has one additional winning coalition than game N: ACDE. According to monotonicity, the voters in the additional winning coalition, ACDE, should have more voting power in M than they do in N, and B should have less voting power in M that she does in N. Examining the Shapley-Shubik power indices for both games, we see that $\frac{66}{120} \geq \frac{1}{2}$ for voter A, $\frac{36}{120} \leq \frac{1}{2}$ for voter B, and $\frac{6}{120} \geq 0$ for voters C, D, and E. This indicates that, on this pair of games, the Shapley-Shubik power index is consistent with the monotonicity property. If we look at the voting powers according to the Banzhaf power index for both games, we see that $\frac{9}{19} \leq \frac{1}{2}$ for voter A, $\frac{7}{19} \leq \frac{1}{2}$ for voter B, and $\frac{1}{19} \geq 0$ for voters C, D, and E. In this example, voter A has less power in game M than in game N, which violates monotonicity. This indicates that, on this pair of games, the Banzhaf power index violates monotonicity. In this way the Shapley-Shubik power index is better than the Banzhaf power index for calculating voting power. The goal of this study is to find all voting power indices that satisfy the properties of normalization, symmetry, dummy, dummy independence, and monotonicity.

3. Three Player Games

To begin working on the problem, I started looking at small games. When looking at three player games, it started getting a little interesting. First, I wrote down all the possible three player games by listing winning coalitions. Since the definition of simple voting games that I am using states that the coalition including all players in the game must be a winning coalition, I began with that as the only winning coalition. Then I added all the possible coalitions with two players to create new games, and then went on to coalitions of one player while making sure I did not have two disjoint winning coaltions. I found five games:

1. ABC
2. ABC,AB
3. ABC,AB,AC
4. ABC,AB,AC,BC
5. ABC,AB,AC,A

By using only the specified properties, you can find the voting power of each voter in each of the 5 games. In the first game, if the voters are relabeled, the winning coalitions don't change. Therefore, based on symmetry, the voters must all have an equal amount of voting power. Since the powers must add up to 1, according to normalization, each voter must have a power of $\frac{1}{3}$. In the second game, voter C can't change a losing coalition to a winning coalition; so by the dummy property, voter C has 0 voting power. By symmetry and normalization, A and B have equal power that must add up to 1 so they must each have a voting power of $\frac{1}{2}$. In the third game, though, it is impossible to find an exact power for the voters. This is because all the voters have some voting power, but they don't all have equal voting power. Voters B and C have equal voting power, but since we don't know the exact value, they are both assigned a value of x. Then since the voting powers must add up to 1, we can assign a value of $1 - 2x$ to voter A. Game four and five are similar to games one and two. In the fourth game, voters A, B, and C all have a voting power of $\frac{1}{3}$. In the fifth game, voter A has a voting power of 1 and voters B and C both have a voting power of 0.

More can be learned about the value of x by using monotonicity. To do this, I compared games 2 and 3. Since voters A and C have one more winning coalition in game 3 than they do in game 2, they need to have more voting power in game 3 than in game 2. Looking at voter A, this shows that $1-2x \geq \frac{1}{2}$ which can be reduced to $x \leq \frac{1}{4}$. Looking at voter C, this shows that $x \geq 0$. Although I still don't have an exact value for x, I know that $0 \leq x \leq \frac{1}{4}$. With this constraint, we have characterized the possible voting powers for three-player simple voting games for any power index that satisfies the five properties.

4. Four Player Games

After looking at three player games, I looked at four player games. I used the same process as I did before to find the voting power of each voter in all games with four voters. Although the three player case was trivial, the four player case became quite complicated. In the following table, I called the voters 1, 2, 3, and 4. Here is a table of my findings:

	Winning Coalitions	Power	Monotonicity Inequalities
1	1234	$\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}$	
2	1234,123	$\frac{1}{3},\frac{1}{3},\frac{1}{3},0$	
3	1234,123,124	$\frac{1}{2}-b,\frac{1}{2}-b,b,b$	2(124): $\frac{1}{3} \leq \frac{1}{2}-b, \frac{1}{3} \geq b, 0 \leq b$
4	1234,123,124,134	$1-3d,d,d,d$	3(134): $\frac{1}{2}-b \leq 1-3d, \frac{1}{2}-b \geq d, b \leq d$
5	1234,123,124,134,234	$\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}$	4(234): $1-3d \geq \frac{1}{4}, d \leq \frac{1}{4}$
6	1234,123,124,12	$\frac{1}{2},\frac{1}{2},0,0$	3(12): $\frac{1}{2}-b \leq \frac{1}{2}, b \geq 0$
7	1234,123,124,134,12	$1-2g-h,h,g,g$	4(12): $1-3d \leq 1-2g-h, d \leq h, d \geq g$
			6(134): $\frac{1}{2} \leq 1-2g-h, \frac{1}{2} \geq h, 0 \leq g$
8	1234,123,124,134,234,12	$\frac{1}{2}-f,\frac{1}{2}-f,f,f$	5(12): $\frac{1}{4} \leq \frac{1}{2}-f, \frac{1}{4} \geq f$
			7(234): $1-2g-h \geq \frac{1}{2}-f, h \leq \frac{1}{2}-f, g \leq f$
9	1234,123,124,134,12,13	$1-2k,k,k,0$	7(13): $1-2g-h \leq 1-2k, h \geq k, g \leq k, g \geq 0$
10	1234,123,124,134,234,12,13	$1-2n-p,n,n,p$	9(234): $1-2k \geq 1-2n-p, k \leq n, 0 \leq p$
			8(13): $\frac{1}{2}-f \leq 1-2n-p, \frac{1}{2}-f \geq n, f \leq n, f \geq p$
11	1234,123,124,134,12,13,14	$1-3m,m,m,m$	9(14): $1-2k \leq 1-3m, k \geq m, 0 \leq m$
12	1234,123,124,134,234,12,13,14	$1-3q,q,q,q$	10(14): $1-2n-p \leq 1-3q, n \geq q, p \leq q$
			11(234): $1-3m \geq 1-3q, m \leq q$
13	1234,123,124,134,234,12,13,23	$\frac{1}{3},\frac{1}{3},\frac{1}{3},0$	10(23): $1-2n-p \geq \frac{1}{3}, n \leq \frac{1}{3}, p \geq 0$
14	1234,123,124,134,12,13,14,1	$1,0,0,0$	11(1): $1-3m \leq 1, m \geq 0$

From the table, we can see that there are fourteen games instead of five, ten variables instead of one, and about thirty monotonicity inequalities instead of two. These inequalities create a convex set in a 10-dimensional space. To get a handle on what this space looks like, I decided to find all the vertices of the convex set. I found an algorithm in [1] to find all the vertices of a polyhedron. I used this algorithm to write a program in Maple. The code for this program can be found in the appendix. Before running the program, I went through the inequalities, getting rid of any that were not as binding as others. For example, $\frac{1}{3} \leq \frac{1}{2}-b$ can be reduced to $\frac{1}{6} \geq b$. From another inequality I knew $b \leq \frac{1}{3}$, but the constraint $b \leq \frac{1}{6}$ is more binding so I eliminated the inequality $b \leq \frac{1}{3}$. I also took out any redundant inequalities. For example, since $b \leq \frac{1}{6}$ and $d \leq \frac{1}{4}$ then $b+d \leq \frac{5}{12}$ which means $b+d \leq \frac{1}{2}$. Therefore the inequality $b+d \leq \frac{1}{2}$ is redundant so I removed it. These were removed to limit degeneracy when running the program. Then I went through and wrote the inequalities in the form *unknowns* $\leq$ *constraints* so they could easily be written as one big matrix. Once I had the inequalities in matrix form, I could then run the program. When I ran the program, though, the computer ended up running out of memory and thus, freezing up. At this point, the program had found 555 vertices and likely had more to find. Already this shows that the space I am dealing with is complex so I decided to go another route.

5. Weighted Order Value

A weighted order value is a variation of the Shapley-Shubik power index. The Shapley-Shubik power index looks at the critical voter in all the orderings of the voters. When the orderings are listed in a vertical fashion, the voters are in columns. The Shapley-Shubik power index considers all the columns to have an equal weight. This means, there is no difference between a voter being critical in column one versus being critical in column two. A weighted order value looks at what happens if those columns have different weights. The weights of each column are denoted as $a_1, a_2, a_3, ..., a_n$ in which the subscript numbers correspond to the column number from left to right. The only way that a voter will be critical in column one is if one voter has all the voting power. In this case, the exact value of voting power for all the voters is known so the weighted order value does not matter. Because of this, a_1 is ignored when looking at weighted order values.

Let $a_2, a_3, \ldots, a_n$ be nonnegative numbers that sum to one. The *absolute weighted order power index* O^a is defined by

$$O_i^a(N, W) = \sum_{k=2}^{n} |\{\pi : pivot(\pi) = i, \pi^{-1}(i) = k\}| a_k$$

I used this approach on the four player games in which the exact voting powers of each of the voters was unknown. I calculated the voting power of each of the voters by writing down all the possible orderings of the four voters and then marking the critical voter in each. I then added up the number of times a voter was critical in each of the columns and multiplied that number by the weight of that column. For each voter, I added those those up and then divided by the total for all the voters. The orderings and values of voting power calculated by using the absolute weighted order power index can be found in the appendix. By using the absolute weighted order power index to calculate voting power in the four player games, each unknown used when calculating power based on the properties, can be written in terms of $a_1, a_2, a_3, ..., a_n$. I substituted these new values for the unknowns into the monotonicity inequalities to see if monotonicity holds for all values of the weights. After doing this, I found these constraints: $a_3(a_2 + 2 + 2a_4) \geq a_2 a_4$, $3a_4 \geq a_3$, and $a_3 + 3a_2 a_4 + 9a_3 a_4 \geq a_3 a_2$. Only weighted order values that satisfy these constraints satisfy monotonicity.

6. Conclusion

The weighted order power index gives a whole new set of power indices to look at. Based on the constaints I found for $a_1, a_2, ..., a_n$ not all of weighted order values satisfy monotonicity. At this point, I am unsure what exactly these constraints mean for the weighted order values. In the future I want to look more at these values and find a way to characterize those that satisfy the constraints, thus satisfying monotonicity. I am hoping to then make a generalization for games with any number of players.

7. References

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