Group Rational/Group Monotonic Allocation Methods on Four Player Convex Games

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Abstract: Three allocation methods for convex cooperative games are described and illustrated with an example. Two properties of allocation methods, group rational and group monotone, are described and illustrated with the example.

1 Definitions

An *n-player cooperative game* is a pair (N, v) where $N = \{1, 2, 3, ...n\}$ is the set of players and v is a real-valued function on all coalitions $S \subset N$ satisfying $v(\emptyset) = 0$.

Example 1 *Let* $N = \{1, 2, 3, 4\}$*,* $v(N) = 100$*,* $v(\{1, 2, 3\} = 78$*,* $v(\{1, 2, 4\}) = 63$*,* $v(\{1, 2\}) = 60$*, and* $v(S) = 0$ *for all other coalitions* S*. This is a* 4*-player cooperative game.*

A game v is *convex* if for all $i \epsilon N$ and $S, T \subseteq N$ with $i \epsilon S \subset T$, we have $v(S) - v(S\backslash\{i\}) \leq v(T) - v(T\backslash\{i\})$. Equivalently, v is convex if for all $S, T \subseteq N$, $v(S) \ + v(T) \leq v(S \cup T) + v(S \cap T)$. Add an explanation as to why example 1 is convex.

A game v is *zero-normalized* if the value of all single player coalitions is equal to zero (0). Formally, a game v is zero-normalized if $v(\{i\}) = 0$ for all $i \in N$. Clearly, example 1 is zero-normalized.

An *allocation* x for the cooperative game (N, v) is a real number payoff x_i to each player $i \in N$ satisfying $\sum_{i \in N} x_i = v(N)$, which is called *efficiency*. An *allocation method* (procedure or value) φ is a function from games to values. For example 1, $x = (40, 30, 20, 10)$, meaning that player 1 would receive 40, player 2 would receive 30, player 3 would receive 20, and player 4 would receive 10, is an allocation for our previous example because efficiency is satisfied $(40 + 30 + 20 + 10 = 100)$.

An allocation x is *group rational* if the allocation gives to each coalition at least as much as they could obtain on their own. Formally, an allocation x is *group rational* if $\sum_{i \in S} x_i \geq v(S)$ for all $S \subseteq N$. Notice that the allocation $(40, 30, 20, 10)$ is group rational for example 1, because: $x_1 + x_2 + x_3 + x_4 = 40 + 30 + 20 + 10 = 100 \geq 100 = v(N)$, $x_1 + x_2 + x_3 = 40 + 30 + 20 = 90 \geq 78 = v(\{1, 2, 3\})$, $x_1 + x_2 + x_4 = 40 + 30 + 10 = 80 \geq 63 = v(\{1, 2, 4\})$, $x_1 + x_2 = 40 + 30 = 70 \geq 60 = v(\{1, 2)\}$, and for all other coalitions S, $\sum_{i \in S} x_i \geq 0 = v(S)$.

An allocation method φ is *group monotone* if the worth of a single coalition increases (decreases), the allocation to the players in that coalition do not decrease (increase). Formally an allocation method φ is *group monotone* if $\varphi_i(u) \leq \varphi_i(v)$, whenever $i \in T, u(T) \leq v(T)$, and $u(S) = v(S)$ for all $S \neq T$.

2 Shapley Value

The Shapley Value**,** Sh, is defined by

$$Sh_i(v) \equiv \sum_{\substack{S \subseteq N \\ S \ni i}} \frac{(|S| - 1)!(|N| - |S|)!}{|N|!} [v(S) - v(S\backslash\{i\})].$$

The Shapley Value for example 1 is calculated in the following chart. In the first column of this chart are the coalitions S that have a non-zero value and the second column are their corresponding worths. In the third column are the calculations on the first part of the Shapley formula. For example, $(4-1)!(4-4)!/4! = 3!/4! = 6/24 = 1/4$. The remaining four columns are the calculations for the second part of the Shapley value. For example, the next-to-last column values are obtained by the calculations: $v(\{1,2,3,4\}) - v(\{1,2,4\}) = 100 - 63 = 37$ and $v(\{1,2,3\}) - v(\{1,2\}) = 78 - 60 = 18$. To evaluate player 3's Shapley Value, multiply the third column numbers by the next-to-last column numbers and add: $37 * 1/4 + 18 * 1/12 = 9.25 + 1.5 = 10.75$.

		(|S|-1)!(|N|-|S|)! / |N|!	v(S)-v(S-i)			
			i			
S	v(S)		1	2	3	4
{1,2,3,4}	100	1/4	100	100	37	22
{1,2,3}	78	1/12	78	78	18	
{1,2,4}	63	1/12	63	63		3
{1,2}	60	1/12	60	60		
All others	0					
			41.75	41.75	10.75	5.75

The allocation is $(41.75, 41.75, 10.75, 5.75)$. Notice that this allocation gives every coalition S as least as much as they are worth, satisfying group rationality.

What if the game changed giving the grand coalition $\{1,2,3,4\}$ ten (10) more, while the values of the others coalitions remain the same. How does our allocation change?

		(|S|-1)!(|N|-|S|)! / |N|!	v(S)-v(S-i)			
			i			
S	v(S)		1	2	3	4
{1,2,3,4}	110	1/4	110	110	47	32
{1,2,3}	78	1/12	78	78	18	
{1,2,4}	63	1/12	63	63		3
{1,2}	60	1/12	60	60		
All others	0					
			44.25	44.25	13.25	8.25

The Shapley Value now yields the allocation $(44.25, 44.25, 13.25, 8.25)$.

Notice that this allocation, too, gives every coalition S as least as much as they are worth, satisfying group rationality. Also notice that when the value of the game increased that the allocation of none of the players decreased, satisfying group monoticity.

3 Dutta Ray

The Dutta ray algorithm can be described as follows:

1. Find the largest per capita worth among all coalitions $\frac{v(S)}{|S|}$, if there are more than one choose the largest per capita with the most players in the coalition.

2. Assign that value to each player $i \in S$, for the largest $\frac{v(S)}{|S|}$,

3. Create a new game on the remaining players by subtracting the assigned values, and

4. Repeat, if necessary.

More formally the Dutta Ray algorithm is,

Step 1: DR: let $v_1 \equiv v$. Then find the unique coalition $S_1 \subseteq N$ such that for all $S \subseteq N$, (i) $\frac{v_1(S_1)}{|S_1|} \geq \frac{v_1(S)}{|S|}$, and (ii) if $\frac{v_1(S_1)}{|S_1|} = \frac{v_1(S)}{|S|}$ and $S \neq S_1$, then $|S_1| > |S|$. Then for all $i \in S_1$,

$$DR_i(v) = \frac{v_1(S_1)}{|S_1|}$$

.

Step k: Suppose that $S_1, ..., S_{k-1}$ have been defined .

Let $N_k \equiv N \backslash (S_1 \cup ... \cup S_{k-1})$. Let v_k be the game for N_k defined by setting for all $S \subseteq N_k$,

$$v_k(S) \equiv v(S_1 \cup ... \cup S_{k-1} \cup S) - v(S_1 \cup ... \cup S_{k-1})$$

Then find the unique coalition $S_k \subseteq N_k$ such that for all $S \subseteq N_k$, (i) $\frac{v_k(S_k)}{|S_k|} \geq \frac{v_k(S)}{|S|}$, and (ii) if $\frac{v_k(S_k)}{|S_k|} = \frac{v_k(S)}{|S|}$ and $S \neq S_k$, then $|S_k| > |S|$. Then for all $i \in S_k$,

$$DR_i(v) = \frac{v_k(S_k)}{|S_k|} = \frac{v(S_1 \cup ... \cup S_k) - v(S_1 \cup ... \cup S_{k-1})}{|S_k|}.$$

Let's calculate the Dutta Ray for the game: The first column of this chart are the coalitions S that have non-zero worths, the second column are their corresponding worths, and the third column are their corresponding per capita worths. The largest per capita worth is 30 corresponding to the coalition $S_1 = \{1, 2\}$.

S	v(S)	v(S)/|S|
{1,2,3,4}	100	25
{1,2,3}	78	26
{1,2,4}	63	21
{1,2}	60	30
All others	0	0

The Dutta Ray algorithm now gives 30 each to player 1 and player 2.

So, now we must calculate the Dutta Ray algorithm for player 3 and player 4. We must evaluate the worth of possible remaining coalition S, that is, we must determine the game on $N_2 = \{3, 4\}$ called v_2 above and called w below.

$w(\{3,4\}) = v(N) - v(\{1,2\})$
$w(\{3,4\}) = 100 - 60$
$w(\{3,4\}) = 40$

$w(\{3\}) = v(\{1,2,3\}) - v(\{1,2\})$
$w(\{3\}) = 78 - 60$
$w(\{3\}) = 18$

$w(\{4\}) = v(\{1,2,4\}) - v(\{1,2\})$
$w(\{4\}) = 63 - 60$
$w(\{4\}) = 3$

As before, we calculate the per capita coalitional worths and find the maximum.

S	w(S)	w(S)|S|
{3,4}	40	20
{3}	18	18
{4}	3	3

Now player3 and player 4 each receive 20. The Dutta Ray algorithm for this game gives us an allocation of $(30, 30, 20, 20)$. Notice that this allocation gives to each coalition S at least as much as they are worth, satisfying group rationality.

Once again, what if the game changed giving the grand coalition $\{1, 2, 3, 4\}$ ten (10) more, while the values of the others coalitions remain the same. How does our allocation change?

S	v(S)	v(S)/|S|
{1,2,3,4}	110	27.5
{1,2,3}	78	26
{1,2,4}	63	21
{1,2}	60	30
All others	0	0

The Dutta Ray still gives 30 to player 1 and player 2. So, again, we must calculate the Dutta Ray for player 3 and player 4.

$w(\{3,4\}) = v(N) - v(\{1,2\})$
$w(\{3,4\}) = 110 - 60$
$w(\{3,4\}) = 50$

$w(\{3\}) = v(\{1,2,3\}) - v(\{1,2\})$
$w(\{3\}) = 78 - 60$
$w(\{3\}) = 18$

$w(\{4\}) = v(\{1,2,4\}) - v(\{1,2\})$
$w(\{4\}) = 63 - 60$
$w(\{4\}) = 3$

S	w(S)	w(S)|S|
{3,4}	50	25
{3}	18	18
{4}	3	3

Now player 3 and player 4 receive 25 each. The Dutta Ray algorithm for this game gives us an allocation of $(30, 30, 25, 25)$. Notice that this allocation gives to each coalition S at least as much as they are worth, satisfying group rationality. Also notice that when the value of the game increased that the allocation of none of the players decreased, satisfying group monoticity.

4 The Robbins Value

A *weak order* on the set of players N is a relation $\preceq$ that is reflexive ($i \preceq i$ for all $i \in N$), complete ($i \preceq j$ or $j \preceq i$ for all $i, j \in N$), and transitive ($i \preceq j$ and $j \preceq k$ implies $i \preceq k$ for all $i, j, k \in N$). We use $i \prec j$ to denote $i \preceq j$ but not $j \preceq i$, and we use $i \sim j$ to denote $i \preceq j$ and $j \preceq i$. A weak order is said to be *strict* if $i \prec j$ or $j \prec i$ for all $i, j \in N$. Let $W(N)$ be the set of all weak orders on N. Let $W'(N)$ be the set of all weak orders that are not strict.For example, $W(\{1,2,3\}) = \{1 \prec 2 \prec 3,\ 1 \prec 3 \prec 2,\ 2 \prec 1 \prec 3,\ 2 \prec 3 \prec 1,\ 3 \prec 1 \prec 2,\ 3 \prec 2 \prec 1,\ 1 \prec 2 \sim 3,\ 2 \prec 1 \sim 3,\ 3 \prec 2 \sim 1,\ 1 \sim 2 \prec 3,\ 1 \sim 3 \prec 2,\ 3 \sim 2 \prec 1,\ 1 \sim 2 \sim 3\}$, and $W'(\{1,2,3\}) = \{1 \prec 2 \sim 3,\ 2 \prec 1 \sim 3,\ 3 \prec 2 \sim 1,\ 1 \sim 2 \prec 3,\ 1 \sim 3 \prec 2,\ 3 \sim 2 \prec 1,\ 1 \sim 2 \sim 3\}$.

The Robbins Value, R, is defined by

$$R_i(v) = \frac{1}{|W'(N)|} \sum_{\preceq \in W'(N)} \frac{v(\{j : j \preceq i\}) - v(\{j : j \prec i\})}{|\{j : j \sim i\}|}$$

For example 1, the notation (12)3 is the same as $1 \sim 2 \prec 3$ and 34(12) is the same as $3 \prec 4 \prec 1 \sim 2$.

	$\frac{v(\{j:j\preceq i\})-v(\{j:j\prec i\})}{\lvert\{j:j\sim i\}\rvert}$				
Weak Order	1	2	3	4	Total
(12)34	30	30	18	22	100
(12)43	30	30	37	3	100
3(12)4	39	39	0	22	100
4(12)3	$31\frac{1}{2}$	$31\frac{1}{2}$	37	0	100
34(12)	50	50	0	0	100
43(12)	50	50	0	0	100
(13)24	0	78	0	22	100
(13)42	0	100	0	0	100
2(13)4	39	0	39	22	100
4(13)2	0	100	0	0	100
24(13)	50	0	50	0	100
42(13)	50	0	50	0	100
(14)23	0	63	37	0	100
(14)32	0	100	0	0	100
2(14)3	$31\frac{1}{2}$	0	37	$31\frac{1}{2}$	100
3(14)2	0	100	0	0	100
23(14)	50	0	0	50	100
32(14)	50	0	0	50	100
(23)14	78	0	0	22	100
(23)41	100	0	0	0	100
1(23)4	0	39	39	22	100
4(23)1	100	0	0	0	100
14(23)	0	50	50	0	100
41(23)	0	50	50	0	100
(24)13	63	0	37	0	100
(24)31	100	0	0	0	100
1(24)3	0	$31\frac{1}{2}$	37	$31\frac{1}{2}$	100
3(24)1	100	0	0	0	100
13(24)	0	50	0	50	100
31(24)	0	50	0	50	100
(34)12	0	100	0	0	100
(34)21	100	0	0	0	100
1(34)2	0	100	0	0	100
2(34)1	100	0	0	0	100
12(34)	0	60	20	20	100
21(34)	60	0	20	20	100

	$\frac{v(\{j:j\preceq i\})-v(\{j:j\prec i\})}{\|\{j:j\sim i\}\|}$				
Weak Order	1	2	3	4	Total
(12)(34)	30	30	20	20	100
(34)(12)	50	50	0	0	100
(13)(24)	0	50	0	50	100
(24)(13)	50	0	50	0	100
(14)(23)	0	50	50	0	100
(23)(14)	50	0	0	50	100
(123)4	26	26	26	22	100
4(123)	$33\frac{1}{3}$	$33\frac{1}{3}$	$33\frac{1}{3}$	0	100
(124)3	21	21	37	21	100
3(124)	$33\frac{1}{3}$	$33\frac{1}{3}$	0	$33\frac{1}{3}$	100
(134)2	0	100	0	0	100
2(134)	$33\frac{1}{3}$	0	$33\frac{1}{3}$	$33\frac{1}{3}$	100
(234)1	100	0	0	0	100
1(234)	0	$33\frac{1}{3}$	$33\frac{1}{3}$	$33\frac{1}{3}$	100
$\frac{Total}{\|W'(N)\|}$	$\frac{1729}{50}$	$\frac{1729}{50}$	$\frac{841}{50}$	$\frac{701}{50}$	5000
	34.58	34.58	16.82	14.02	100

So the Robbins value gives us the allocation $(34.58, 34.58, 34.58, 14.02)$. This allocation is group rational.

So what if we change the value of the grand coaltion {1,2,3,4} once again to 110. How would that change affect our allocation?

	$\frac{v(\{j:j\preceq i\})-v(\{j:j\prec i\})}{\|\{j:j\sim i\}\|}$				
Weak Order	1	2	3	4	$Total$
(12)34	30	30	18	32	110
(12)43	30	30	47	3	110
3(12)4	39	39	0	32	110
4(12)3	31.5	31.5	47	0	110
34(12)	55	55	0	0	110
43(12)	55	55	0	0	110
(13)24	0	78	0	32	110
(13)42	0	110	0	0	110
2(13)4	39	0	39	32	110
4(13)2	0	110	0	0	110
24(13)	55	0	55	0	110
42(13)	55	0	55	0	110
(14)23	0	63	47	0	110
(14)32	0	110	0	0	110
2(14)3	31.5	0	47	31.5	110
3(14)2	0	110	0	0	110
23(14)	55	0	0	55	110
32(14)	55	0	0	55	110
(23)14	78	0	0	32	110
(23)41	110	0	0	0	110
1(23)4	0	39	39	32	110
4(23)1	110	0	0	0	110
14(23)	0	55	55	0	110
41(23)	0	55	55	0	110

Weak Order	1	2	3	4	$Total$
	$\frac{v(\{j:j\preceq i\})-v(\{j:j\prec i\})}{\lvert\{j:j\sim i\}\rvert}$				
(24)13	63	0	47	0	110
(24)31	110	0	0	0	110
1(24)3	0	31.5	47	31.5	110
3(24)1	110	0	0	0	110
13(24)	0	55	0	55	110
31(24)	0	55	0	55	110
(34)12	0	110	0	0	110
(34)21	110	0	0	0	110
1(34)2	0	110	0	0	110
2(34)1	110	0	0	0	110
12(34)	0	60	25	25	110
21(34)	60	0	25	25	110
(12)(34)	30	30	25	25	110
(34)(12)	55	55	0	0	110
(13)(24)	0	55	0	55	110
(24)(13)	55	0	55	0	110
(14)(23)	0	55	55	0	110
(23)(14)	55	0	0	55	110
(123)4	26	26	26	32	110
4(123)	$36\frac{2}{3}$	$36\frac{2}{3}$	$36\frac{2}{3}$	0	110
(124)3	21	21	47	21	110
3(124)	$36\frac{2}{3}$	$36\frac{2}{3}$	0	$36\frac{2}{3}$	110
(134)2	0	110	0	0	110
2(134)	$36\frac{2}{3}$	0	$36\frac{2}{3}$	$36\frac{2}{3}$	110
(234)1	110	0	0	0	110
1(234)	0	$36\frac{2}{3}$	$36\frac{2}{3}$	$36\frac{2}{3}$	110
$\frac{Total}{\lvert W'(N)\rvert}$	$\frac{1854}{50}$	$\frac{1854}{50}$	$\frac{966}{50}$	$\frac{826}{50}$	$\frac{5500}{50}$
	37.08	37.08	19.32	16.52	110

The Robbins value for this game gives us an allocation of $(37.08, 37.08, 19.32, 16.52)$. Notice that this allocation gives to each coalition S at least as much as they are worth, satisfying group rationality. Also notice that when the value of the game increased that the allocation of none of the players decreased, satisfying group monoticity.

5 Conclusion

The Shapley value was introduced in Shapley (1953). It was proved to be group rational on convex games in Shapley (1971). The Dutta Ray Algorithm was first introduced by Hokari (2002), which is equivalent to the original Dutta Ray solution introduced by Dutta and Ray (1989). Hokari proved in his paper that the Dutta Ray algorithm is group monotone. By definition the original Dutta Ray is group rational so by definition of equivalence the Dutta Ray algorithm is group rational. This is the introduction of the Robbins value. It is my conjecture that the Robbins value is group rational and group monotone on four player convex games.

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7 References

1. Shapley, Lloyd S. [1953], "A Value for n-Person Games," *Contributions to the Theory of Games*, vol. II, Ann. Math. Studies 28, Princeton University Press, Princeton, New Jersey, pp.307-317.

2. Shapley, Lloyd S. [1971], "Cores on Convex Games," *International Journal of Game Theory*, Volume 1, pp.11-26.

3. Dutta, Bhaskar and Ray, Debraj [1989], "A Concept of Egalitarianism Under Participation Constraints," *Econometrica*, Volume 57, No. 3, pp.615-635.

4. Hokari, Toru [2002], "Monotone-path Dutta-Ray solutions on convex games," *Social Choice and Welfare*, Volume 19, pp.825-844.