Intergenerational Games

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In certain cultures, decisions are made based on their effects on future generations. Rather than seeking ephemeral gains for themselves, policy makers at both the government and family levels project into the future and speculate as to how their decisions will effect their progeny. An extreme case is that of the Iroquois indians who ask "how will this decision effect our ancestors seven generations from now?" when making tribal decisions. Thoughts of these situations as well as realizations of recent global environmental trends has led me to consider a new class of cooperative games, games in which one player, the present decision maker, controls the game and the fate of the other players who are subjected to the abuses as well as the sacrifices of the dominant player.

My research involves what I will refer to as intergenerational games; that is, the players are the decision makers of each generation. I will analyze the effects of projecting into the future and considering the welfare of following generations in the decisions made by the present generation. My analysis will begin with the case of allocating a fixed, non-renewable resource with consideration for zero, one or two generations into the future. After presenting a specific instance, I will then generalize these results and prove that the Shapley value allocates an unattainable amount to the initial generation except under special circumstances. The next step in this research will be to analyze the case of renewable resources, the notion of overlapping generations, and competition within each generation.

The following assumptions will hold for the specific example and the general case which follows.

1. There exists a fixed amount, R_0 units, of some desirable resource, and a population of size N which consumes this resource.
2. Now or in the near future, scarcity of this resource will be a problem.
3. There is a decreasing per capita utility of using the resource.

4. There is an increasing marginal cost of using the resource.
5. Both the population and the resource have a zero growth rate. (this will be relaxed later on)

Define $U(r)= k_1(r/N)^{1/a}$: The utility for using r units of R_0

$C(r)= k_2(r/R_0)^b$: The cost of using r units of R_0

$P(r)= U(r) - C(r)$: The benefit of using r units

Taking a derivative of P(r) and setting it equal to zero yields the following result:

Formula 1: $$r^* = \left(\frac{k_1\ R_0^{\ b}}{k_2\ a\ b\ N^{1/a}}\right)^{a/(ab-1)}$$

The second derivative test indicates that r_* is a local maximum. Now I assign the value of the coalition g_i to be $P(r_i^*)$.

Example 1:

Let a=b=2 $R_0 = 1000$; $N = 50$; $k_1 = 1$; $k_2 = 2$;

Formula 1 gives $r_{1*} = 679$.
Hence $v(g_1)= 3.685 - 0.922 = 2.763$
For g_2 $R_1 = 321$, hence formula 1 gives $r_{2*} = 149$
thus, $v(g_2)= 1.726- 0.431 = 1.295$

Determining $v(g_1g_2)$ is more difficult.
I want to maximize the joint benefit; that is,

$$\text{Max: } (r_1/50)^{1/2} + (r_2/50)^{1/2} - 2(r_1/1000)^2 - 2(r_2/(1000 - r_1))^2$$
$$\text{s.t. } r_1 + r_2 \leq 1000$$
$$r_1, r_2 \geq 0$$

Dynamic programming yields the following numerical results:
$r_1^* = 375$
$r_2^* = 363$
thus, $v(g_1g_2) = 2.739 + 2.694 - 0.281 - 0.675 = 4.477$

Finally, the Shapley value for g_1 and g_2 are:

$= 2.973$
$= 1.504$

At this point a few observations are in order.

First of all, when working alone; that is, with only its own interests in mind, the first generation receives a benefit of 2.763 while consuming 679 units of R_0. However, when considering the second generation, the initial group only receives a benefit of 2.458 and consume only 375 units. Moreover, while they would receive less by forming the coalition, the Shapley value indicates that they should receive more. This seems reasonable as the first generation is sort of 'sacrificing' its own welfare for its kindred. However, one other point needs to be made by this example before we move on.

The Shapley value assigns the first generation a value of 2.973. This amount is impossible to attain under any circumstances. When the first generation is maximizing its own welfare we determined that at most it could obtain a benefit of 2.763. Even if they tried to, they could never reach the Shapley value. This is where the intergenerational games begin to distinguish themselves from standard cooperative games; as no side payments can be made from the second generation to the first, the second generation reaps a tremendous benefit at the expense of the first. Why then, would a society willingly sacrifice its own benefit for its progeny? Perhaps some people equate their children's benefit with their own; perhaps people feel that as their parents sacrificed for them that they must do so as well. The overlapping generations model will attempt to explain this difficulty; however, before moving on to a more complex model, I thought it best to observe a two generational projection and focus on the middle generation which both benefits from its parents' sacrifices and foregos some of its own gain for the sake of its children.

For the values of the singleton players, I simply maximized the benefit of each generation based on what the preceding generations had left; formula 1 yields these numbers as R_0 is readjusted for the second and again for the third player. For the value of two successive generations, I used the dynamic programming approach as explained in the simpler model, again adjusting R_0 accordingly. The benefits of the odd players were determined by allowing them to simply maximize their own gains without consideration for the others. In order to assign the value to the coalition of the first and third generation I used the following equations:

$$\text{Max: } r_1/50 + r_3/50 - 2(r_1/1000) - 2(r_3/(1000-r_1-r_2))$$

$$\text{s.t. } r_1 + r_2 + r_3 = 1000$$

$$r_1, r_2, r_3 = 0$$

$$\text{where } r_2 = (1000-r_1)^{4/3} / (4*50^{1/3}) \text{ as given by formula 1.}$$

This maximizes the joint value of the first and third generations knowing that the second will act only to maximize its own gain.

To determine the value of the grand coalition, I again used the dynamic programming approach except I maximized the sum of the net gains for all three generations. Using the same numbers as in the single projection example, I determined the following values for all possible coalitions:

	$P[g_1]$	$P[g_2]$	$P[g_3]$
$v(g_1) = 2.763$			
$v(g_2) = 1.295$			
$v(g_3) = 0.855$			
$v(g_1 g_2) = 4.477$	2.458	2.019	1.131
$v(g_2 g_3) = 2.223$	2.763	1.262	0.961
$v(g_1 g_3) = 3.705$	2.698	1.656	1.007
$v(g_1 g_2 g_3) = 5.850$	2.390	1.900	1.570

Thus, the Shapley values are:

$\phi(v) = (3.135, 1.660, 1.055)$

Notice again how the first generation receives less than its Shapley value no matter what. In some coalitions, the middle generation attains more than the Shapley value and in others it receives less. The third generation always receives more when a coalition is formed; even when it is not in the coalition.

This leads to my first theorem.

Lemma: Intergenerational games are superadditive.

Proof: Consider any coalition of players. By simply using their r^* values, they ensure a coalition value which will equal the sum of their individual values. Possibly, they can attain more through reallocating the resource, but they must get at least that amount.

Theorem 1: In an intergenerational game with zero growth, the Shapley value allocates an attainable amount to the initial generation if and only if the game is strictly additive with respect to the first generation.

Proof: Given that the game is strictly superadditive with respect to at least one coalition containing the first player, we have:

$$= q_1[v(g_1)-v(o)] + q_2[v(g_1 g_2) - v(g_2)] + \ldots +$$

$$q_n[v(g_1g_2...g_n)-v(g_2g_3...g_n)]$$

$$> q_1[v(g_1)] + q_2[v(g_1)+v(g_2)- v(g_2)] + ... + q_n[v(g_1)+ v(g_2g_3...g_n)- v(g_2g_3... g_n)]$$

$$= q_1[v(g_1)] + q_2[v(g_1)] + ... + q_n[v(g_1)]$$

$$= v(g_1) \text{ as } \quad q \quad = 1.$$

Thus, the Shapley allocation to the first generation is strictly greater than the maximum value, $v(g_1)$, which can be obtained. Only when the game is additive with respect to every coalition containing the first player will the allocation be feasible.

Theorem 2: $v(g_1g_2) \not> v(g_1) + v(g_2)$ if and only if $r^{1*} = R_0$.
Proof: I will just give an informal argument as the algebra gets messy...

part1 (show $v(g_1g_2) = v(g_1) + v(g_2)$ when $r_1^* = R_0$)
Given $r_1^* = R_0$, then $v(g_1) = k_1(r_1/N)^{(1/a)} - k_2$ and $v(g_2)=0$. Maximize the joint benefit of the coalition containing only the first two players. The result shows $v(g_1g_2) = v(g_1) + v(g_2)$.

part2 (show $r_1^* = R_0$ when $v(g_1g_2) = v(g_1) + v(g_2)$)
Given $v(g_1g_2) = v(g_1) + v(g_2)$, we know that the same r values which maximize the joint benefit also maximize the individual benefits. Plugging r_1^* and r_2^* into the equations which maximize the joint benefit and solving yields the desired result.

Thus, $v(g_1g_2)$ is strictly greater than $v(g_1) + v(g_2)$ if and only if $r_1^* = R_0$.

Conjecture: The only time in which these games are strictly additive, and thus the only time when the Shapley value can be obtained is when $r_{1*} = R_0$.

The next phase in my research was to observe the case in which the generations are overlapping. In this model, side payments are possible; however, other complications arise.

Consider the case in which the generations are overlapping; that is, at a given period, three generations are present with the middle one being the decision maker. What should the middle generation do? When it was the youngest generation, it probably received some utility from the preceding generation and thus 'owes' something to it. However, it must also provide some utility to the younger generation as a form of security for its old age. How should the middle generation allocate the available utility with these considerations in mind?

As in the preceding case, the first problem is where to start. Initiating the problem at some random point creates a dummy player; that is, the oldest generation can not add any value to a coalition and hence has no worth. By most allocation methods, this generation should not receive any utility. Consider the preceding case in which players were valuing their children's worth as their own.

The Shapley value allocated 2.973 units of utility to the first generation and 1.504 to the second (consider the older generation as the zeroeth which is allocated 0 units.) The problem with the non-overlapping generations case was that only 2.458 units were available in the first period which made it impossible for the Shapley value to be attained. Now it is possible; however, a fair allocation method is necessary to distribute this utility.

The first generation deserves all that it can attain in the first period and more. Thus, one consideration would be to give it the entire amount in the first period and then have the second generation make up the difference in the second period. This seems unreasonable as the second generation would seem unmotivated to provide the first with anything when given the opportunity later on. Therefore, one ought to provide the next group with at least something.

Another possibility is to give each generation their proportional share of the Shapely value in each period, regardless of whomever is in charge of distributing the utility.
In this case the younger generation always receives less than the older one, even when they are in charge of the distribution.

In any case, a few guidelines seems clear. First of all, the primary obligation of the middle generation is to repay its debt to the older generation. Next, it should allocate some utility to the younger generation; however, they should never give them so much that they themselves will be unable to be paid back in the next time period. With these criterion, the allocation of resources seems dependent on the particular culture involved.

When considering the effect of one's decisions on the future, at least in the case of resource allocation, it seems evident that cooperation is key in increasing the aggregate benefits over many generations. By initiating a future based decision making society, the current generation must sacrifice its own gain. However, once the society has begun such a program, both the individual gains and the total gains will increase. Considering the possibility of overlapping generations allows the initial generation to retrieve its lost utility at a later time period, but it also necessitates a fair allocation scheme in which different amounts of utility must be distributed at different times to players who are not always around. While this research may not have led to any dramatic results, the ideas contained within are important and might perhaps lead to something of consequence at a later time, whether it is accomplished by myself or another....