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Math 300

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Sprouts: More Than Vegetables

There are many games which have mathematical bases. Checkers, chess, and hex are some good examples. One such game is called Sprouts. Sprouts consists of two players using lines to connect points in such a way that they force their opponent to become "stuck". An analysis of this game leads to several conjectures that can be made about game play. The first step, however, is to carefully say what is meant by the phrase "a game of Sprouts."

To begin, a description of the rules and goals of the game is in order. The game starts with a set of points, usually a small number like three or four, in a plane. The first player draws a line segment of any shape which starts on a point and ends on a point. The line segment can start and end on the same point, or start and end on different points. After the segment is drawn, the player must place a new point somewhere on the segment. Now player two must do the same: draw a line segment which starts on a point and ends on a point, and then place a new point on the newly drawn segment. The only conditions which exist are that no one point may have more than three connections to it, and lines cannot cross each other. Players continue taking turns until one of them cannot make another move. That player loses.

The phrase "a game of Sprouts" will refer to the entire process, from the first move by the first player, to the point at which one of the players cannot make another move. Now that the rules of play have been clearly established, conjectures can be formulated.

A logical question about Sprouts would be: does every game have a winner? By realizing that there can be no ties (since the first player unable to go immediately loses), the question becomes: does every game end? The following proof answers that question.

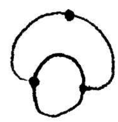
Conjecture: *A game of Sprouts will always end.*

Proof: Consider a game of Sprouts which begins with an arbitrary number of points n. The number n is clearly a positive integer. We will define the term "connection node" to mean an imaginary place on a point at which a line segment can begin or end. Since each point can have at most three connections to it, the number of total possible connection nodes is $3n$. After the first player draws a line segment, two of those connections are used up. By placing a new point on that segment, three new possible connections are created, two of which are used up immediately, since the point is placed on the segment. Therefore each segment drawn uses up two connection nodes, but only creates one new connection node, for a net loss of one connection node. Since at the end of a turn, a new connection node is created, there will always be at least one connection node in play, even if the next player cannot make a move. This means that the number of nodes in a game will always be ≥ 1. So it is clear that if t is the number of turns in the game, $3n-t \geq 1$, since one connection node is lost each turn. Therefore the number of turns is given by $t \leq 3n-1$. Since $3n-1$ is a distinct positive integer, the number of turns cannot be infinite, and each game must end at some point.

So now it is known a game of Sprouts will always end. Since there can be no ties, there must always be a winner. The next logical question is: Assuming both players are trying to win, can either player assure a win in a game of Sprouts? That is a very good, very difficult question. Begin by examining a game of sprouts which begins with only one point.

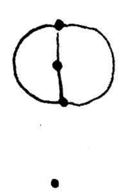
Player 1 has only one choice. He must draw a segment which starts and ends on that point, and place a new point somewhere on that line.

Player 2 has two pseudo-choices which are really only one choice. Player 2 must connect the two points drawn by either going across the loop created by Player 1 or by going around the loop created by Player 1. Either way, Player 2 uses up the only two remaining connection nodes, and creates one, leaving Player 1 only one connection node to work with. This means that Player 1 cannot go, and he loses.

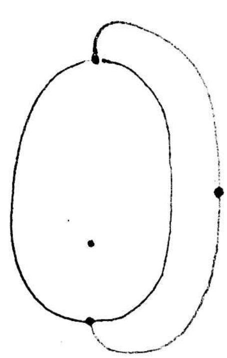


So in a 1-point game of Sprouts, Player 2 must win. But can Player 2 always win? Examine now a 2-point game of Sprouts.

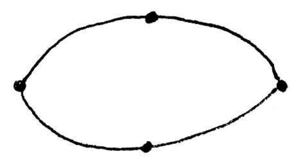
Player 1 has three choices: either start and end on the same point without encircling the other point, start and end on the same point encircling the second point, or start and end on different points. In the first case, Player 2 needs only to connect the new point drawn by Player 1 with the point Player 1 started and ended on inside the loop to ensure a win, since the new set-up is equivalent to a one-point game.



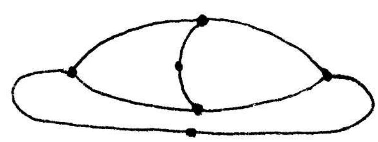
In the second case, Player 2 needs only to connect the new point drawn by Player 1 with the point Player 1 started and ended on outside the loop to ensure a win, since the new set-up is equivalent to a one-point game.

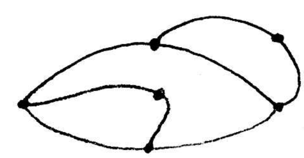


The final case is a bit more complicated. Player 1 has begun by connecting the two points in some way, and placing a new point on the drawn segment. Player 2 connects the same two points that Player 1 connected, and places his new point on the drawn segment.



There are now a total of four connection nodes left; one for each point. If Player1 chooses to connect any two of the points inside the "loop" that exists, Player 2 needs only to connect the other two outside the "loop" to ensure a win. If Player 1 connects any two of the points outside the "loop," then Player 2 needs only to connect the other two points inside the "loop" to ensure a win. Player 2 in essence "counters" each move of Player 1.





Things are looking good for Player 2. He can ensure a win in both the 1-point and 2-point scenarios. However, an examination of a three-point game is much more involved than that which has been discussed above. However, since we now know that Player 2 can win any 1-point or 2-point game, it seems logical that in order to win any other game, Player 2 would need only to make moves such that the new set-up is equivalent to a 1-point or 2-point game. But is this possible? That is a question beyond the scope of any present information that has been gathered. From some preliminary study, it is clear however, that at least for some 3-point game scenarios, Player 2 can indeed ensure a win, making it clear that if either player has the ability to ensure a win in any game of Sprouts, it must be Player 2.