Tom Ventrudo & Jodi Wallman   Final Report: August 11, 1991

FINDING A VALUE ON PARTIALLY DEFINED GAMES

Introduction

A cooperative game is a pair (N, v) where N = {1, 2, ..., n} and v is a real-valued function on the nonempty subsets of N. Elements of N are often called players while subsets of N are called coalitions. The grand coalition, denoted as N, is composed when all of the players cooperate together. The function v is called the worth function, and v(S) is interpreted as the worth of the coalition S. In other words, v(S) is the amount that individuals in S can jointly obtain if they cooperate as a group.

A game (N, v) is said to be <u>superadditive</u> if

$v(S \cup T) \geq v(S) + v(T)$ for all disjoint coalitions S, T $\subseteq$ N. The game (N, v) is <u>monotonic</u> if $v(S) \leq v(T)$ for all coalitions S $\subseteq$ T $\subseteq$ N. The game (N, v) is said to be <u>0-normalized</u> if v(i) = 0 for all i $\in$ N. The <u>0-normalization</u> of a game (N, v) is the game (N, u) where

$$u(S) = v(S) - \sum_{i \in S} v(i) .$$

A game (N, v) is <u>0-monotonic</u> if its 0-normalization is monotonic.

In an effort to find some standard of fairness or a predictor of bargaining solutions, many researchers of game theory have concentrated on finding the "best" way for individuals in a game to form coalitions and eventually maximize their savings. Once the savings have been made they must be allocated to the participating players. An <u>allocation method</u> is a function that assigns an

allocation to each cooperative game in some class.

For every cooperative game there are $2^N - 1$ possible coalitions. Normally the worth of every coalition is known. However, this may not always be the case. For instance, a firm might have certain time or monetary constraints that prevent it from knowing the worth of all coalitions. The utility for a game of this type occurs when it is impractical to determine every coalitional worth, $v(S)$. During the summer of 1990, while involved in a Research Experience for Undergraduates program at Drew University, David Letscher developed and began to research what might happen in such cases. He named this topic partially defined games.

Necessary definitions

A <u>partially defined game</u>, or PDG, is a triple (N, Z, v) where $N = \{1, 2, \ldots, n\}$ is the set of players, Z is a collection of nonempty subsets of N, and v is a real-valued function on Z. We will often use v to denote a partially defined game when N and Z are clear from context.

<u>Definition</u> -   Let J be a subset of the player set N such that 1, $n \in J$. A <u>J-game</u> is a partially defined game where $Z = \{S \subseteq N \mid |S| \in J\}$.

We will denote a J-game by (N, J, v). Z is the set of coalitions whose worths are known, while J is the set of sizes, or cardinalities, of coalitions whose worths are known. For example, in a 4 - player partially defined game where the worths of the grand coalition, the triples, and the singletons are known while the worths of the pairs are not, $J = \{1, 3, 4\}$. For this same game, $Z = \{A, B, C, D, ABC, ABD, ACD, BCD, ABCD\}$. In our research,

we concentrated on 0-normalized games like this, where J = {1, n-1, n}.

An extension of the partially defined game (N, Z, v) is a cooperative game (N, $\hat{v}$) satisfying $\hat{v}$(S) = v(S) for all S $\in$ Z. A partially defined game (N, Z, v) is superadditive, monotonic or 0-monotonic if there exists an extension that is superadditive, monotonic, or 0-monotonic respectively. A value is a function from partially defined games to payoff vectors in $\mathbb{R}^n$.

A Complete Exploration of the 4 - Player Game

Introduction

By restricting this portion of the paper to a 4 - player game in which J = {1, 3, 4}, we hope to clarify the ways in which one might allocate the savings of a partially defined game. First, we must determine possible worths for every coalition, even those which are not in Z. The following is the list of worths for the general case of such a game, where v(S)=0 when |S|=1. Note: a≥b≥c≥d≥e≥0:

S	v(S)	S	v(S)
ABCD	a	AB	?
ABC	b	A C	?
AB D	c	A D	?
A CD	d	BC	?
BCD	e	B D	?
		CD	?

Clearly, the minimum possible worth for each pair is zero. By using superadditivity, we can estimate the maximum of the range of values for all of the pairs. For example, v(ABC) ≥ v(AB) + v(C). We know that v(C) = 0, so v(ABC) = b ≥ v(AB). By a similar argument, v(ABD) = c ≥ v(AB). Thus, 0 ≤ v(AB) ≤ c. Repeat this

process for every unknown worth until the following range points
are known:

S	v(S)	S	v(S)
ABCD	a	AB	0 - c
ABC	b	A C	0 - d
AB D	c	A D	0 - d
A CD	d	BC	0 - e
BCD	e	B D	0 - e
		CD	0 - e

We can view the game extension where $\hat{v}$ = (AB, AC, AD, BC, BD, CD) in a coordinate system in which the space is determined by the number of unknown worths. For example, here the game space is in six dimensions, because we have six unknown worths. The real game lies within a certain probability distribution of the extensions of the game v. One frequently used allocation method is the Shapley value, Φ. The Shapley value for a player i is the average of the marginal values player i brings to the group of all individuals over possible orderings. It can be calculated using the following formula, where s denotes $|S|$:

$$\phi_i(N, v) = \sum_{S \in N} (((s-1)!(n-s)!)/n!)[v(S) - v(S-i)]$$

In determining an allocation for this game, we look at the expected value of the Shapley value of the game extensions. The Shapley value is linear so we obtain the following:

$$E(\Phi(\hat{v})) = \Phi(E(\hat{v})).$$

The allocation obtained when using a uniform probability distribution results from finding the centroid of the game extensions and taking its Shapley value. If the figure in the game space is a regular figure, such as a cube or a rectangular box,

then the centroid is the average of the vertices of the figure. This approach is discussed below. Otherwise, a weighted probability distribution may be used, and we take a weighted average of the vertices of the figure in the game space.

Boxes, Centroids, and Extensions

In the 4 - player example above, we look at the 0-monotonic extensions of the game v. The vertices of the figure in the game space can be characterized by placing either of the two range points in each of the six coordinates, forming a convex set. For each of the six coordinates there are two choices, for example in the first coordinate you may place 0 or c, therefore creating $2^6 = 64$ vertices. These vertices form a rectangular box in which the centroid is (c/2, d/2, d/2, e/2, e/2, e/2). We take the Shapley value of the centroid and call it the __internal value__ (Iota). It is defined as $I(v) = \Phi$ (centroid of $\hat{v}$). Finding the Shapley value for the players in the above game, yields the following internal value:

$$I_A = a/4 + b/12 + c/8 + d/6 - 3e/8$$
$$I_B = a/4 + b/12 + c/8 - d/3 + e/8$$
$$I_C = a/4 + b/12 - 7c/24 + d/12 + e/8$$
$$I_D = a/4 - b/4 + c/24 + d/12 + e/8$$

The Superadditive Cases

The following is a look at the 4 - player game where superadditive extensions are used rather than 0-monotonic ones. Look at the superadditive extensions of the game below:

S	v(S)	S	v(S)
ABCD	a	AB	0 - c
ABC	b	A C	0 - d
AB D	c	A D	0 - d
A CD	d	BC	0 - e
BCD	e	B D	0 - e
		CD	0 - e

By superadditivity, v(ABCD) $\geq$ v(AB) + v(CD), therefore,

a $\geq$ c + e. Similarly a $\geq$ d + e. Recall that c $\geq$ d $\geq$ 0 from above,

so a $\geq$ c + e is a sufficient condition. There are three cases of

superadditive extensions defined by these conditions.

Case I. a $\geq$ c + e. This condition forces the superadditive

extensions to be equivalent to the 0-monotonic ones. The vertices

are found, the centroid of the box is computed, and the internal

value is placed in accordance with the above section.

Case II. a < c + e and a $\geq$ d + e. Because the restriction of

c and e has been lifted the vertices take on a different set of

values. Namely, (0, 0 or d, 0 or d, 0 or e, 0 or e, 0),

$$(0, ..., ..., ..., ..., e),$$
$$(c, ..., ..., ..., ..., 0),$$
$$(c, ..., ..., ..., ..., a-c),$$
$$\text{and } (a-e, ..., ..., ..., ..., e).$$

All possible permutations of the above values form a cut-off box in

6-dimensions with 80 vertices. A cross-section of the box in two-

dimensions (x,y) is shown below:
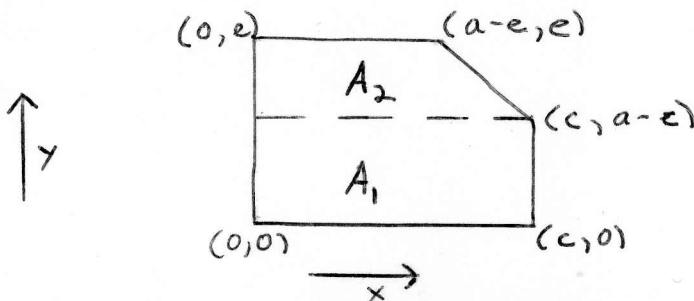
In computing the centroid of the box we first find the area of

the figure: Area = $A_1 + A_2$.
$$A_1 = l \times w = c(a-c) = ac - c^2.$$
$$A_2 = 1/2(b_1 + b_2)h = 1/2(a-e + c)(e - (a-c))$$
$$= 1/2(c^2 + 2ae - a^2 - e^2)$$
Therefore, Area = $ac + ae - 1/2(c^2 + a^2 + e^2)$.

Assume the centroid is in the rectangular portion of the box. Find

the value along the x-axis that balances half the area of the box

by solving $A/2 = ex$ for x to obtain

$x = (2ac + 2ae - c^2 - a^2 - e^2)/4e$; call this value x1. Repeat this process in the y-direction by solving $A/2 = cy$ for y to get $y = (2ac + 2ae - c^2 - a^2 - e^2)/4c$; call this value y1. Show that x is in the box by showing $x \leq a - e$. This inequality results in $(a-c)^2 \geq 3e^2 - 2ae$. Similarly for $y \leq a - c$, $(a-e)^2 \geq 3c^2 - 2ac$. If the centroid is not in the rectangular portion of the box, then one or both of the inequalities is reversed. If $x > a - e$, then solve the quadratic $A/2 = (1/2)(2a - c-x)(c-x)$ for x, where the right hand side is the area of the trapezoid formed by dropping a perpendicular line at x. Solving for x yields

$x = a \pm (\sqrt{2}/2)(\sqrt{(a^2 - 2ac + 2ae + c^2 - e^2)}$. Disregard $a + (\sqrt{2}/2)(\sqrt{(a^2 - 2ac + 2ae + c^2 - e^2)}$ because it does not lie in the original trapezoid. Similarly if $y > a - c$, then $y = a - (\sqrt{2}/2)(\sqrt{(a^2 - 2ae + 2ac - c^2 + e^2)}$. Call these values x2 and y2 respectively. Once the centroid has been found, a new game extension has been created and one can easily take its Shapley value to produce the internal value for this game.

<u>Summary of the Procedure in Case II.</u>

The following is a pseudo-algorithm for computing the centroid:

    If $(a-c)^2 \geq 3e^2 - 2ae$
        Then use x1.
        Else use x2.

    If $(a-e)^2 \geq 3c^2 - 2ac$
        Then use y1.
        Else use y2.

This process always results in the centroid of the cross-section in two-dimensions, because the remaining four dimensions look the same

as you look out from the cross-section. For example, when the centroid of the above figure is in the rectangle, its coordinates are:

$$((2ac+2ae-c^2-a^2-e^2)/4e,d/2,d/2,e/2,e/2,(2ac+2ae-c^2-a^2-e^2)/4c).$$

Case III. $a < d + e$. In this case, the restrictions have been lifted on all of the pairs, resulting in a six-dimensional box that is cut off in all directions. There are 125 vertices to this box, because each of the five permutations used above for c are used here for c, d, and d again. This results in $5^3 = 125$ vertices. The centroid is computed by using the algorithm above, and the Shapley value of the game using the centroid of the game extensions is taken to produce the internal value.

Conclusion

In a 4-player partially defined game, the internal value can be used to allocate savings. The above procedures show that this value can be determined regardless of what type of extensions are used. The procedures also work when $J = \{1,2,4\}$. Therefore, a value exists on all 4-player games, but it is yet to be determined if the internal value generalizes to n players.

Generalizing the Shapley Value on Partially Defined Games

Introduction

In a partially defined game, some of the v(S) in the Shapley value formula will be unknown. As shown above, by using the superadditivity property, one can estimate the ranges of the missing values. These form the extensions of the game, which are a convex set of values. In order to find which value might be the

best one to use, we find the vertices of our convex set.

A function for finding the vertices

When we find the vertices of the set of possible values in the manner stated earlier, points which do not even exist in our set sometimes appear to be vertices. For example, look at the following general case of the 5 - player partially defined game where $J=\{1,4,5\}$, $v(S)=0$ when $|S|=1$, and $a \geq b \geq c \geq d \geq e \geq f \geq 0$:

S	v(S)	S	v(S)
ABCDE	a	BC E	0 - f
ABCD	b	B DE	0 - f
ABC E	c	CDE	0 - f
AB DE	d	AB	0 - v(ABE)
A CDE	e	A C	0 - v(ACE)
BCDE	f	A D	0 - v(ADE)
ABC	0 - c	A E	0 - v(ADE)
AB D	0 - d	BC	0 - v(BCE)
AB E	0 - d	B D	0 - v(BDE)
A CD	0 - e	B E	0 - v(BDE)
A C E	0 - e	CD	0 - v(CDE)
A DE	0 - e	C E	0 - v(CDE)
BCD	0 - f	DE	0 - v(CDE)

If we find the vertices of this set as we did before, it would be possible for $v(ABC) = 0$ while $v(AB) = d > 0$. By superadditivity, though, this can never be the case.

In addition, some of the vertices actually coalesce, forming single vertices which should intuitively hold more weight than others. For example, the vertices of the 5 - player game above form a set which looks like a box on top of another box in 20-dimensional space. If we set $v(ACDE) = v(BCDE) = 0$, the vertices form a set in which the top box shrinks to a single point. Thus, eight vertices collapse into one.

Rather than estimate all the ranges of v(S) for all S where $|S| \notin J$, find the 2^N possible vertices, and then throw away all of the vertices which cannot exist and add in those which occur more

than once, we have defined a function for finding the vertices of any partially defined game:

Given $\delta(S)$ where $\delta: 2^N \rightarrow \{0,1\}$.
Define $\hat{v}_\delta(S) = \delta(S)\min\{v(T) \mid S \subseteq T\} + (1 - \delta(S))\max\{\hat{v}(R) \mid R \subseteq S\}$.

In other words, we can randomly assign every coalition which has an unknown worth a $\delta(S) = 0$ or 1. There will be 2^N such combinations. For each combination, we can go down the list of coalitions of unknown worths and find $\hat{v}_\delta(S)$ for each such coalition. Thus, each sequence, δ, will yield a point, $\hat{v}_\delta$, in the monotonic extensions of the partially defined game. Henceforth, the monotonic extensions of the partially defined game, v, shall be denoted as ME(v). Below, we show that each of these points is a vertex.

<u>Theorem 1</u>: Given any partially defined game, (N, Z, v), $\hat{v}_\delta$ is a vertex of the monotonic extensions of v.

<u>Proof</u>:

If for all $\hat{v}_1$ and $\hat{v}_2 \in$ ME(v) such that $\hat{v}_1 + \hat{v}_2 = 2\hat{v}_\delta$ implies that $\hat{v}_1 = \hat{v}_2 = \hat{v}_\delta$, then $\hat{v}_\delta$ is a vertex.
Assume $\hat{v}_1$ and $\hat{v}_2 \in$ ME(v) satisfy $\hat{v}_1 + \hat{v}_2 = 2\hat{v}_\delta$.

Base Case - Let $|S| = 2$.
If $\delta(S) = 0$, then $\hat{v}_\delta(S) = \max\{\hat{v}_\delta(R) \mid R \subseteq S\}$, by definition of $\hat{v}_\delta(S)$.
$\qquad\qquad\qquad = \max\{v(R) \mid R \subseteq S\}$, because $|S|=2$.
$\qquad\qquad\qquad = 0$, because worth of all singletons is 0.
$\qquad$ Thus, $\hat{v}_1(S) + \hat{v}_2(S) = 0$.
$\qquad$ $\hat{v}_1(S) \geq 0$ and $\hat{v}_2(S) \geq 0$, by monotonicity.
$\qquad$ Therefore, $\hat{v}_1(S) = \hat{v}_2(S) = \hat{v}_\delta(S)$.
If $\delta(S) = 1$, then $\hat{v}_\delta(S) = \min\{v(T) \mid S \subseteq T\}$, by definition.
$\qquad\qquad\qquad \geq \hat{v}_1(S), \hat{v}_2(S)$, by monotonicity.
$\qquad$ But, $\hat{v}_1(S) + \hat{v}_2(S) = 2\hat{v}_\delta(S)$, by assumption.
$\qquad$ Therefore, $\hat{v}_1(S) = \hat{v}_2(S) = \hat{v}_\delta(S)$.

Inductive Hypothesis - Assume $\hat{v}_1(S) = \hat{v}_2(S) = \hat{v}_\delta(S)$, where $|S| \leq g$.

Inductive Step - Prove $\hat{v}_1(S) = \hat{v}_2(S) = \hat{v}_\delta(S)$, where $|S| = g+1$.
If $\delta(S) = 0$, then $\hat{v}_\delta(S) = \max\{\hat{v}_\delta(R) \mid R \subseteq S\}$, by definition of $\hat{v}_\delta(S)$.
$\qquad\qquad\qquad = \hat{v}_\delta(R)$ for some $R \subseteq S$, where $|R| \leq g$.
$\qquad$ Thus, $\hat{v}_1(S) + \hat{v}_2(S) = 2\hat{v}(R)$.
$\qquad$ $\hat{v}_1(S) \geq \hat{v}_1(R) = \hat{v}(R)$ and $\hat{v}_2(S) \geq \hat{v}_2(R) = \hat{v}(R)$, by
$\qquad\qquad$ monotonicity and the inductive hypothesis.
$\qquad$ Therefore, $\hat{v}_1(S) = \hat{v}_2(S) = \hat{v}_\delta(S)$.
If $\delta(S) = 1$, then $\hat{v}_\delta(S) = \min\{v(T) \mid S \subseteq T\}$, by definition.

$$\geq \hat{v}_1(S), \; \hat{v}_2(S), \text{ by monotonicity.}$$
$$\text{But, } \hat{v}_1(S) + \hat{v}_2(S) = 2 \, \hat{v}_\delta(S), \text{ by assumption.}$$
$$\text{Therefore, } \hat{v}_1(S) = \hat{v}_2(S) = \hat{v}_\delta(S).$$

Hence, all $\hat{v}_1$ and $\hat{v}_2 \in ME(v)$ satisfying $\hat{v}_1 + \hat{v}_2 = 2 \, \hat{v}_\delta$ imply that $\hat{v}_1$ = $\hat{v}_2 = \hat{v}_\delta$, and thus $\hat{v}_\delta$ is a vertex.

Therefore, given any partially defined game, (N, Z, v), $\hat{v}_\delta$ is a vertex of the monotonic extensions of v.∎

In order to use the vertices to find an allocation for our partially defined game, it is important that we know all of them. How can we be certain that we have found them all? Is it possible that the 2^N different δ yield the exhaustive list of vertices?

<u>Conjecture</u>: If $\hat{v}$ is a vertex of ME(v), then $\exists \; \delta$ such that $\hat{v} = \hat{v}_\delta$.
<u>Sketch of argument</u>:
Suppose $\nexists \; \delta$ such that $\hat{v} = \hat{v}_\delta$.
Thus, $\exists \; S$ such that $\delta(S) \neq 0$ or 1, and
$\quad \min\{v(T) \mid S \subseteq T\} > \hat{v}(S) > \max\{\hat{v}(R) \mid R \subseteq S\}$.
Show $\hat{v}$ is not a vertex of ME(v).
We believe that if $\hat{v}(S) \neq \min\{v(T) \mid S \subseteq T\}$ or $\max\{\hat{v}(R) \mid R \subseteq S\}$, then it cannot be a vertex because there will be points on either side which average to the "vertex."
Thus, $\hat{v}$ will not be a vertex.
So, if $\hat{v}$ is a vertex of ME(v), then $\exists \; \delta$ such that $\hat{v} = \hat{v}_\delta$.∎

<u>From Centroids to Averages</u>

After using the above function to find all of the vertices for a partially defined game, we can attempt to find the centroid of the set. Then, we could take the Shapley value of this point and thus have found an allocation for the game. The centroid seems a strong candidate for the "best" value, but finding the centroid of game extensions of games with more than four players is not an easy task. Therefore, the centroid does not seem to be a practical solution. In our quest for a better value, we next turned to a simple average of the vertices.

Finding an average is a common event in mathematics. The problem is that if one of the values being averaged is far away from the others, then the average could be skewed. An important

question was whether such a problem arises in partially defined games, or are the games such that this average value is actually a fair allocation method? As discussed earlier, the average of the vertices of the monotonic extensions of a 4 - player partially defined game is, in fact, the centroid. With games of more than four players, though, the sets lack noticeable symmetry, and the average is skewed away from the centroid.

The Weighted Average

Instead of averaging the vertices, we use a weighted probability distribution to take a weighted average of the vertices of the figure in the game space. A definite pattern to the weighted average exists. Once the weighted average for each coalition of unknown worths is found, we can find the Shapley value for this extension.

To find the weighted average of the vertices, one can use the following formula, where s is any coalition of unknown worth:

$$\hat{v}(S) \quad \hat{v} = \sum_{\substack{R \subseteq S \\ R \notin Z}} \frac{1}{2^{\alpha(R,S)}} \min\{v(T) \mid R \subseteq T\}$$

where $\alpha(R,S)$ = some integer between 1 and n - 1. The alpha function is determined by the value of each supercoalition, T,

$$\min\{v(T) \mid R \subseteq T\}$$

where $R \subseteq T$. If we arrange the v(T) in descending order, we can then assign $\alpha(R,S)$ in ascending order from 1 to n - 1. Thus, the greatest v(T) is multiplied by the smallest $\alpha(R,S)$, and the smallest v(T) is multiplied by the greatest $\alpha(R,S)$. Some examples of the weighted average of the vertices for coalitions not in Z are given below:

$$\hat{v}(ABC) = c/2 + d/4 + e/8 + f/16$$
$$\hat{v}(ABD) = d/2 + d/4 + e/8 + f/16 = 3d/4 + e/8 + f/16$$
$$\hat{v}(BCD) = f/2 + f/4 + f/8 + f/16 = 15f/16$$
$$\hat{v}(AB) = d/2$$

Now that we can find the weighted average of the vertices for every partially defined game without having to find every vertex and then take the average, we would like to be able to do the same for the Shapley value. We have detected a pattern to the Shapley value at the weighted average of the vertices, but as yet are unable to generalize it. The Shapley value for the general cases of both the 4 - and 5 - player games are listed below. Following those examples, we have given our observations:

4 - players:
$$\Phi_A = a/4 + b/12 + c/8 + d/6 - 3e/8$$
$$\Phi_B = a/4 + b/12 + c/8 - d/3 + e/8$$
$$\Phi_C = a/4 + b/12 - 7c/24 + d/12 + e/8$$
$$\Phi_D = a/4 - b/4 + c/24 + d/12 + e/8$$

5 - players:
$$\Phi_A = a/5 + b/20 + c/15 + 2d/15 + 9e/40 - 19f/40$$
$$\Phi_B = a/5 + b/20 + c/15 + 2d/15 - 59e/160 + 19f/160$$
$$\Phi_C = a/5 + b/20 + c/15 - 17d/60 + 23e/480 + 19f/160$$
$$\Phi_D = a/5 + b/20 - 9c/40 + d/120 + 23e/480 + 19f/160$$
$$\Phi_E = a/5 - b/5 + c/40 + d/120 + 23e/480 + 19f/160$$

Observations:
1. The sum of these values is a = v(ABCD) and a = v(ABCDE), respectively.
2. For every player, a fraction of one of the worths is subtracted. In every case, that worth is the value of the coalition not Z of which the player under consideration is NOT a member.
3. Every player gets $1/|N|$ of the grand coalition.
4. If we factor $1/|N|$ out of every value, we see the following:

4 - players:
$$\Phi_A = 1/4 \ (a + b/3 + c/2 + 2d/3 - 3e/2)$$
$$\Phi_B = 1/4 \ (a + b/3 + c/2 - 4d/3 + e/2)$$
$$\Phi_C = 1/4 \ (a + b/3 - 7c/6 + d/3 + e/2)$$
$$\Phi_D = 1/4 \ (a - b + c/6 + d/3 + e/2)$$

5 - players:

$$\Phi_A = 1/5 \ (a + b/4 + c/3 + 2d/3 + 9e/8 - 19f/8)$$
$$\Phi_B = 1/5 \ (a + b/4 + c/3 + 2d/3 - 59e/32 + 19f/32)$$
$$\Phi_C = 1/5 \ (a + b/4 + c/3 - 17d/12 + 23e/96 + 19f/32)$$
$$\Phi_D = 1/5 \ (a + b/4 - 9c/8 + d/24 + 23e/96 + 19f/32)$$
$$\Phi_E = 1/5 \ (a - b + c/8 + d/24 + 23e/96 + 19f/32)$$

Notice that $1/|N-1|$, $1/|N-2|$, and $2/3$ appear often in the above.

Conclusion

Although we do not have a tangible result for the n - player case, we believe that one does exist. When it became clear that the centroid approach, would not generalize to n - players, we turned to averages as a more accessible method. In the future, we hope that someone may be able to generalize and build upon the work which we have already done.

References

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