# The Bargaining Problem: John Nash's Mathematical Vision

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## The Bargaining Problem

- Find a partner and determine who is player A and who is player B.
- Come to a written and signed agreement as to how to split the $10 and bag of chocolate within 4 minutes.
- If an agreement is not reached within 4 minutes, person A gets $4 and person B gets nothing.
- Only one pair will actually get the money and chocolate. They must be willing to answer a few questions publicly.
- No external threats or enticements are permitted.
- Any questions?
- Begin negotiating!

## The Bargaining Game

- John Nash's first insight: Rather than the possible distributions of money and chocolate, determine the feasible utility pairs.
- Utility is a model for player choices.
- Everyone should answer the following question on their index card.
- For what amount of money, $a_i$ dollars, would you be indifferent between receiving the money or receiving the bag of chocolate?

# Utility Model 1

- Assume that player $i$'s utility for receiving $m$ dollars and proportion $c$ of the bag of chocolate is given by
$$u_i(m, c) = m + a_i c.$$

- The set of *feasible* utility pairs is
$$\{(u_1(m, c), u_2(10 - m, 1 - c)) : 0 \leq m \leq 10, 0 \leq c \leq 1,$$
and the disagreement utility pair is
$$(u_1(4, 0), u_2(0, 0)).$$

- For our simple utility function assumption, the feasible utility pairs form a quadrilateral with vertices $(10 + a_1, 0)$, $(10, a_2)$, $(0, 10 + a_2)$, and $(a_1, 10)$, and the disagreement utility pair is $(4, 0)$.

## Nash Properties

- A *solution method* assigns a feasible utility pair to each bargaining game.
- A utility pair is *rational* if each player receives at least their disagreement utility..
- A solution method is ...
- *efficient* if it assigns efficient utility pairs: it is impossible to increase the utility of one player without decreasing the utility of the other player.
- *unbiased* if it gives the same utility to each player of a game in which the disagreement point is $(0, 0)$ and $(x, y)$ is rational if and only if $(y, x)$ is rational.
- *scale invariant* if, whenever a new game is obtained by a positive linear rescaling of a player's utilities, the assigned utility pair for the new game is the same positive linear rescaling of the assigned utility pair for the original game.
- *independent of irrelevant utility pairs* if removing feasible utility pairs other than the assigned utility pair does not change the assigned utility pair.

## Nash Theorems

- If a solution method is efficient, unbiased, and scale invariant, then for any game whose rational utility pairs form a filled right triangle, the method assigns the midpoint of the hypotenuse.
- There exists a unique efficient, unbiased, scale invariant, and independent of irrelevant utility pairs solution method. This solution method chooses the rational utility pair $(x, y)$ that maximizes $(x - d_1)(y - d_2)$.
- Nash's Theorems are general. We need to ensure that our utility functions faithfully model player choices.

# Utility Model 2

- Assume that player $i$'s utility for receiving $m$ dollars and proportion $c$ of the bag of chocolate is given by

$$u(m, c) \;=\; \underbrace{10\left(\tfrac{m}{10}\right)^{\mu}}_{\text{money utility}} \;+\; \underbrace{a\,c^{\gamma}}_{\text{chocolate utility}}, \qquad \mu, \gamma > 0.$$

- For what probability $b$ would you be Indifferent between receiving (1) a lottery in which you win \$10 with probability $b$, and (2) \$5?
- Indifference yields $b \cdot 10 = 10(0.5)^{\mu} \Rightarrow \mu = \dfrac{\ln b}{\ln(0.5)}$.
- For what probability $p$ would you be Indifferent between receiving (1) a lottery in which you win the entire bag of chocolate with probability $p$, and (2) half of the bag of chocolate?
- Indifference yields $p\,a = a(0.5)^{\gamma} \Rightarrow \gamma = \dfrac{\ln p}{\ln(0.5)}$.

# Utility Model 3

- The next level of sophistication would be to relax the separability of money and chocolate assumption, but there will not be sufficient time to try this.
- Nash, John F., Jr. (1950). “The Bargaining Problem.” Econometrica, 18(2), 155–162. https://doi.org/10.2307/1907266.
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