# Game Theoretic Modeling for Math Majors

## A Biological Auction

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August 2019

# A Strange Auction

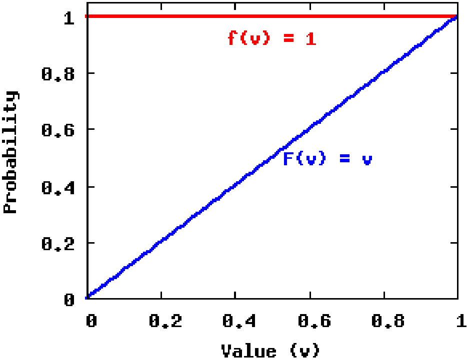
- Open ascending bid auction for a prize.
- The highest bidder wins the prize but pays her bid.
- The second highest bidder wins nothing but pays his bid.
- No one else pays.
- Play now!
- Biological interpretation.

# A Strange Auction Repeated

- The value of the prize to you is on the paper and was drawn from a uniform distribution on 0 to 1000.
- Sealed (nonnegative) bid auction for the prize.
- Both of us pay the lower bid, but only the higher bidder wins the prize.
- Repeat up to 30 times with a variety of opponents.
- Keep track of the strategy you use and its effectiveness.
- Play now!
- What were the most effective strategies?

# Strange Auction Model

- Both players pay the lower bid, but only the higher bidder wins the prize.
- A player knows what the prize is worth to him/her but not what it is worth to his/her opponent.
- $f(v)$ is the probability density the prize is worth $v$ to a player.



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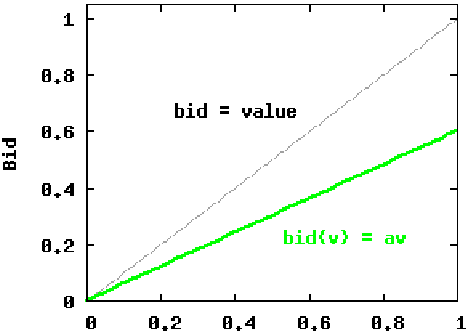
- $\beta(v)$ is the opponent's bid if the prize is worth $v$ to him.



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- $\beta(v)$ is the opponent's bid if the prize is worth $v$ to him.

- If I value the prize at $v$ and bid $b$, my expected payoff is

$$\pi(b) = \int_{\beta(u)<b} (v - \beta(u))f(u)\,du - b\int_{\beta(u)\geq b} f(u)\,du$$

- Assume $\beta(v)$ is the player's payoff maximizing bid, that is,

$$\pi(\beta(v)) \geq \pi(b)$$

for all $b \geq 0$.

# Payoff Maximization (General Case)

- Maximize the following at $b = \beta(v)$:

$$\pi(b) = \int_{\beta(u)<b} (v - \beta(u))f(u)\,du - b\int_{\beta(u)\geq b} f(u)\,du$$

- Take the derivative.

$$\pi'(b) = vf(\beta^{-1}(b))/\beta'(\beta^{-1}(b))) - (1 - F(\beta^{-1}(b)))$$

- First order necessary condition $\pi'(\beta(v)) = 0$.

$$0 = \; vf(v)/\beta'(v) - (1 - F(v))$$

- Solve for $\beta'$.

$$\beta'(v) = \frac{vf(v)}{1 - F(v)}$$

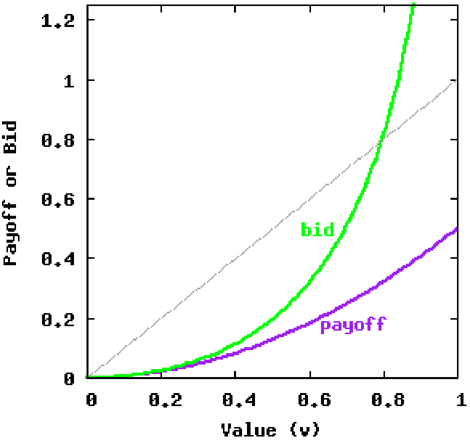
- Solve for $\beta$.

$$\beta(v) = \int_0^v \frac{uf(u)}{1 - F(u)}\,du$$

- This function is differentiable and increasing from $\beta(0) = 0$.

# Payoff Maximization (Special Case)

- Suppose $f(u) = 1, u \in [0,1]$ and $F(u) = u, u \in [0,1]$.
- $\beta(v) = \int_0^v \frac{uf(u)}{1-F(u)}\, du = \int_0^v \frac{u}{1-u}\, du = -v - \ln(1-v)$.
- $\pi_{\max}(v) = \frac{1}{2} v^2$.



# Payoff Maximization Verification

- To verify we have found a maximum, substitute

$$\beta'(v) = \frac{vf(v)}{1 - F(v)}$$

- into
$$\pi'(b) = vf(\beta^{-1}(b))/\beta'(\beta^{-1}(b))) - (1 - F(\beta^{-1}(b)))$$
- to obtain
$$\pi'(b) = (1 - F(\beta^{-1}(b))(v/\beta^{-1}(b) - 1)$$
- which is positive if $b < \beta(v)$
- and negative if $b > \beta(v)$.

# Payoff Using the Strategy

- The payoff to a player who values the prize at $v$ and bids $b$

$$\pi(b) = \int_0^{\beta^{-1}(b)} (v - \beta(u))f(u)\,du - b(1 - F(\beta^{-1}(b)))$$

- is maximized at $b = \beta(v) = vf(v)/(1 - F(v))$

$$\pi_{\max}(v) = \int_0^{v} (v - \beta(u))f(u)\,du - \beta(v)(1 - F(v))$$

- Hence,

$$\pi_{\max}(0) = 0$$

- Taking the derivative

$$\begin{aligned}
\pi'_{\max}(v) &= (v - \beta(v))f(v) + F(v) - \beta'(v)(1 - F(v)) + \beta(v)f(v) \\
&= vf(v) + F(v) - \frac{vf(v)}{1 - F(v)}(1 - F(v)) \\
&= F(v) \geq 0
\end{aligned}$$

- The more you value the prize, the higher your expected payoff.

# Surprising Observation

- Recall the optimal bidding strategy.

$$\beta(v) = \int_0^v \frac{uf(u)}{1 - F(u)}\, du$$

- Find the average bid.

$$\int_0^\infty \beta(v) f(v)\, dv = \int_0^\infty \int_0^v \frac{uf(u)}{1 - F(u)} du \; f(v)\, dv$$

- Interchange integrals ($0 \le u \le v < \infty$).

$$\int_0^\infty \beta(v) f(v)\, dv = \int_0^\infty \frac{uf(u)}{1 - F(u)} \int_u^\infty f(v)\, dv\, du$$

- Since the inner integral is $1 - F(u)$,

$$\int_0^\infty \beta(v) f(v)\, dv = \int_0^\infty uf(u)\, du$$

- The average bid equals the average value.
- For some prize values $v$, the bid $\beta(v)$ is greater than the value!

# Ideas and Practices

- Strategic game with incomplete information
- Interesting questions remain unanswered (e.g., will the Nash equilibrium strategies arise in an evolutionary model?)
- Active student involvement
- Use of prizes

# Minicourse Conclusions

- Observations? Questions? Take aways?
- Game theory is a great topic for a course exhibiting the use of undergraduate mathematics for modeling real-world phenomena.
- It is important to play the games and engage in modeling.
- Material is available to support this approach.
- Instructor engagement and creativity is both productive and fun.

# The Last Slide

- Slides: www.goshen.edu/dhousman under presentations sections, click on *Game Theoretic Modeling for Math Majors*
- Email: dhousman@goshen.edu, rick.gillman@valpo.edu
- Assessment
- Thank you!