Game Theoretic Modeling for Math Majors

Coalition Games

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EPA

The Environmental Protection Agency (EPA) has mandated improvements in the sewage treatment facilities in the cities of Avon, Barport, Claron, and Delmont. Each city could work separately, but $140 million would be saved by all four working together. Some smaller groups of cities also can save money:

Coalition	ABCD	ABC	ABD	ACD	AB	other
Savings	140	108	96	84	24	0

For cooperation to occur and the savings to be obtained, there must be a written and signed agreement among the cities stating how the savings is to be allocated among the cities. With no agreement, a city saves nothing ($0 million).

Play the game and share observations!

A Very Simple Game

Coalition	Gain
ABCD	100
other	0

What is the fair allocation in this game?

Fairness dictates that each player should receive 25.

Definition

An allocation is *efficient* if it is impossible to increase the payoff of one player without decreasing the payoff to another player.

Definition

An allocation is *unbiased* if players who are distinguishable only by their names are allocated the same payoff.

A Somewhat Simple Game

Coalition	Gain
ABCD	150
ABC	150
other	0

What is the fair allocation in this game?

Fairness dictates that D should receive 0 and each other player should receive 50.

Definition

An allocation is *subsidy free* if players that never contribute to or detract from gains are allocated zero.

A Not So Simple Game

Coalition	Gain
ABCD	250
ABC	150
other	0

What is the fair allocation in this game?

Unbiased implies that players A, B, and C should receive the same amount, but it is unclear how much D should receive.

A Not So Simple Game

Coalition	Very Simple Gain		Somewhat Simple Gain		Not So Simple Gain
ABCD	100	+	150	=	250
ABC	0	+	150	=	150

Player	Very Simple Payoff		Somewhat Simple Payoff		Not So Simple Payoff
A, B or C	25	+	50	=	75
D	25	+	0	=	25

Definition

An allocation method for coalition games is *additive* if whenever a coalition game is the sum or difference of other coalition games, the allocation for the original coalition game is the corresponding sum or difference of the allocations of the other coalition games.

Properties Applied to the EPA Game

Coalition	G1		G2		G3		G4		G5		EPA
ABCD	24	+	84	+	72	+	84	−	124	=	140
ABC	24	+	0	+	0	+	84	−	0	=	108
ABD	24	+	0	+	72	+	0	−	0	=	96
ACD	0	+	84	+	0	+	0	−	0	=	84
AB	24	+	0	+	0	+	0	−	0	=	24
other	0	+	0	+	0	+	0	−	0	=	0

Player	A1		A2		A3		A4		A5		EPA
A	12	+	28	+	24	+	28	−	31	=	61
B	12	+	0	+	24	+	28	−	31	=	33
C	0	+	28	+	0	+	28	−	31	=	25
D	0	+	28	+	24	+	0	−	31	=	21

Shapley Allocation Method

Theorem

The Shapley allocation method is the unique allocation method that is efficient, unbiased, subsidy free, and additive.

Order	Marginal Contribution			
	A	B	C	D
ABCD	0	24 - 0 = 24	108 - 24 = 84	140 - 108 = 32
ABDC	0	24 - 0 = 24	140 - 96 = 44	96 - 24 = 72
⋮	⋮	⋮	⋮	⋮
BACD	24 - 0 = 24	0	84 - 0 = 84	140 - 108 = 32
BADC	24 - 0 = 24	0	140 - 96 = 44	96 - 24 = 72
⋮	⋮	⋮	⋮	⋮
DCAB	84 - 0 = 84	140 - 84 = 56	0 - 0 = 0	0
DCBA	140 - 0 = 140	0 - 0 = 0	0 - 0 = 0	0
Average	1464/24 = 61	792/24 = 33	600/24 = 25	504/24 = 21

Shapley Allocation Method Objection #1

Coalition	ABCD	ABC	ABD	ACD	AB	other
EPA	140	108	96	84	24	0
Cost Overrun	120	108	96	84	24	0

Player	A	B	C	D
EPA	61	33	25	21
Cost Overrun	56	28	20	16

The coalition ABC can obtain 108 on their own, but the Shapley allocation only gives $56 + 28 + 20 = 104$!

Definition

An allocation is *group rational* if each coalition receives at least its worth.

Shapley Allocation Method Objection #2

EPA Game and Its Shapley Allocation

Coalition	ABCD	ABC	ABD	ACD	AB	other
Savings	140	108	96	84	24	0

Player	A	B	C	D
Shapley	61	33	25	21

Reduced Game and Its Shapley Allocation

Coalition	AB	A	B
Savings	61+33 = 94	84-25-21 = 38	0

Player	A	B
Shapley	38+56/2 = 66	0+56/2 = 28

Definition

An allocation method is *consistent* if the allocation for any reduced game is the same as the allocation for the original game.

Prenucleolus Allocation Method

Theorem

The prenucleolus is the unique allocation method that is efficient, unbiased, scale invariant, and consistent. Moreover, the prenucleolus is group rational whenever there is a group rational allocation.

Definition

An allocation method is *scale invariant* if (1) multiplying all savings by the same number M results in the method multiplying all payoffs by M, and (2) if changing all savings containing a player i by the same amount B results in the method changing player i's payoff by B.

Definition

The *excess* of a coalition at an allocation is the amount received by the coalition less its worth. The *prenucleolus* is the efficient allocation that successively maximizes the smallest excesses.

Prenucleolus Allocation Method

Coalition	Allocation to A		B		C		D		Worth		Excess
ABC	74	+	28	+	22			−	108	=	16
ABD	74	+	28			+	16	−	96	=	22
ACD	74			+	22	+	16	−	84	=	28
BCD			28	+	22	+	16	−	0	=	66
AB	74	+	28					−	24	=	78
AC	74			+	22			−	0	=	96
AD	74					+	16	−	0	=	90
BC			28	+	22			−	0	=	50
BD			28			+	16	−	0	=	44
CD					22	+	16	−	0	=	38
A	74							−	0	=	74
B			28					−	0	=	28
C					22			−	0	=	22
D							16	−	0	=	16

Prenucleolus Allocation Method Objection

Coalition	ABCD	ABC	ABD	ACD	AB	other
EPA	140	108	96	84	24	0
Cost Overrun	120	108	96	84	24	0

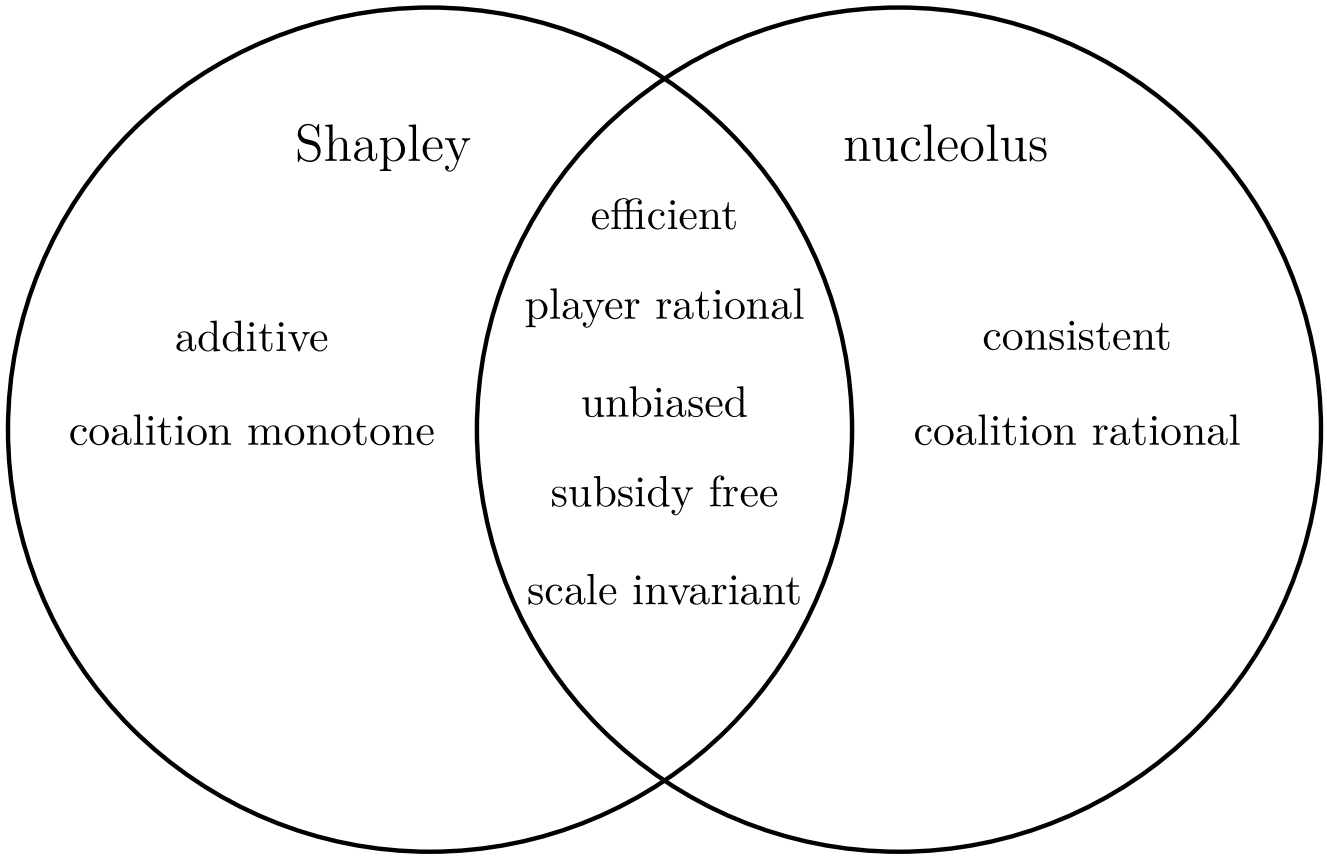
Player	A	B	C	D
EPA	74	28	22	16
Cost Overrun	84	18	12	6

A cost overrun results in player A receiving more!

Definition

An allocation is *group monotone* if increases (decreases) in a single coalitions worth do not result in a decrease (increase) in the payoff for any player who is in the coalition.

Property Comparison



Theorem

There is no allocation method for coalition games with four or more players that is efficient, rational, and coalition monotone.

Ideas and Practices

- Coalition games
- Identify assumptions before methods
- Use algorithms and deductive arguments
- Use properties to compare methods
- Active student involvement