Game Theoretic Modeling for Math Majors

Constructing Utility Functions

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Introduction

- Game Theoretic Modeling for Math Majors
- David Housman, Goshen College, dhousman@goshen.edu
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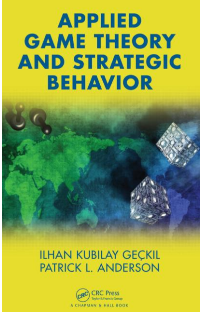
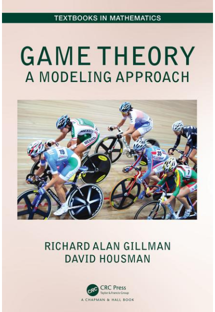
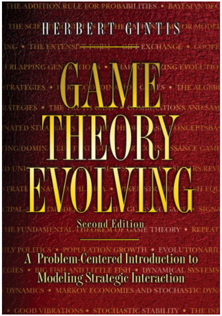
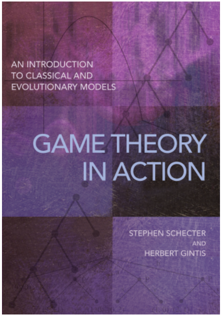
- Who are you? Why are you here? What do you hope this get out of this minicourse?
- Outline of four hours:
    1. Constructing utility functions
    2. Strategic games
    3. Coalition games
    4. A biological auction
- Slides: www.goshen.edu/dhousman under presentations sections, click on *Game Theoretic Modeling for Math Majors*

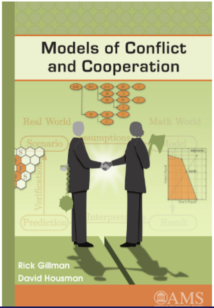
Some Relevant Texts

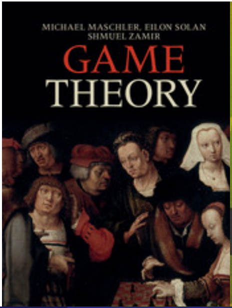


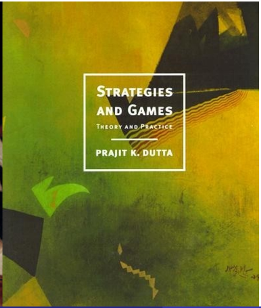




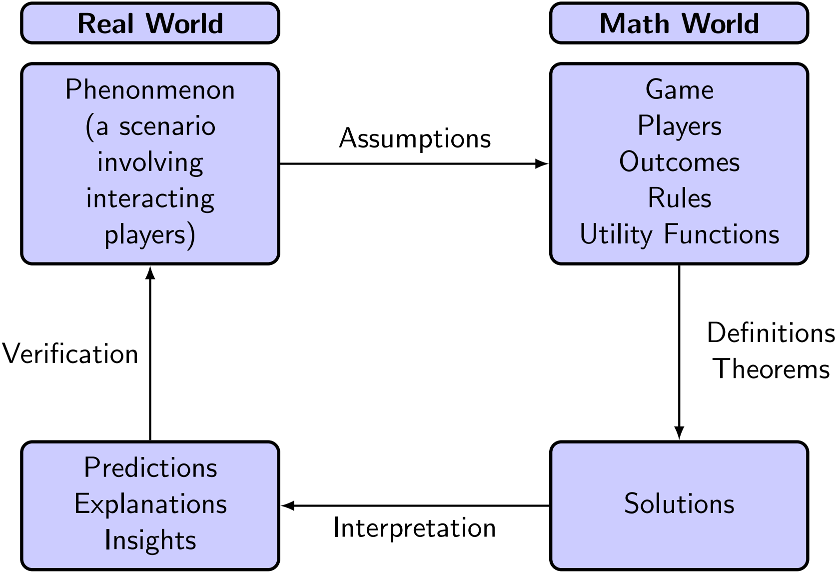








Game Theoretic Modeling Process



Modeling Player Preferences I

- Ordinal utility functions are used to model a player's choices among subsets of outcomes.
- Example outcomes: chocolate, book, cards, \$10 gift, and nothing.
- Here is the construction process.
  1. Set the current set to contain all five outcomes.
  2. Choose the most preferred outcome from the current set.
  3. Remove the outcome chosen from the current set.
  4. If the current set is nonempty, go to step 2.
  5. Assign strictly decreasing utilities to the outcomes in the order chosen.
- Verification requires checking on the choices from all other sets of outcomes (finite but exponential in the number of outcomes).

Ordinal Preferences Characterization

Definition

A player is said to have *ordinal preferences* among outcomes if there exists a utility function u from the set O of outcomes into the real numbers, $\mathbb{R}$, such that whenever presented with a subset $O' \subseteq O$ of outcomes, the player chooses any of the outcomes that maximize u over all outcomes $o \in O'$.

Definition

A player's pairwise choices are *complete* if a (weak) choice can always be made; *transitive* if whenever A is chosen over B and B is chosen over C, then A is chosen over C; and *generalizable* when A is chosen from O' if and only if A is chosen over B for each outcome $B \in O'$.

Theorem

A player has ordinal preferences over a finite set of outcomes if and only if the player's pairwise choices are complete, transitive, and generalizable.

Modeling Player Preferences II

- Interval scale utility functions are used to model intensity of player preferences by considering choices among lotteries of outcomes.
- An example set of outcomes: chocolate, book, cards, $10 gift, and nothing.
- Partially construct and verify a von Neuman-Morgenstern (vNM) utility function for a volunteer participant.
- Each participant, on an index card, should partially construct and verify their own vNM utility function with the assistance of a partner.
  1. Choose the most preferred outcome A from the set of all outcome, and assign $u(A) = 1$.
  2. Choose the least preferred outcome B from the set of all outcome, and assign $u(B) = 0$.
  3. For each remaining outcome C, determine the probability p such that the player is indifferent between C and $pA + (1 - p)B$, and assign $u(C) = p$.
- Verification requires checking on the choices from all other sets of outcomes and lotteries (infinite!).

von Neumann-Morgenstern Preferences Characterization

Definition

A player is said to have *vNM preferences* among outcomes O if there exists a function $u : \mathcal{L} \to \mathbb{R}$ from the set of lotteries $\mathcal{L}$ into the real numbers, $\mathbb{R}$, satisfying the following conditions:

1. The player has ordinal preferences among lotteries modeled by u.
2. Expected Utility Hypothesis: $u_i(p_1 o_1 + \ldots + p_m o_m) = p_1 u_i(o_1) + \ldots + p_m u_i(o_m)$ for any lottery $p_1 o_1 + \ldots + p_m o_m \in \mathcal{L}$.

Theorem

A player has vNM preferences over the lotteries $\mathcal{L}$ associated with a finite set of outcomes and a vNM utility function $u : \mathcal{L} \to \mathbb{R}$ satisfying the Expected Utility Hypothesis if and only if the player's pairwise choices within $\mathcal{L}$ are complete, transitive, generalizable, probabilistic, monotonic, continuous, and substitutable.

von Neumann-Morgenstern Preferences Characterization

Definition

A player's pairwise choices are *probabilistic* if the player is indifferent between any two lotteries that have the same probability distribution over the original outcome set; *monotonic* if whenever the player chooses A over B and $0 \leq p < q \leq 1$, the player chooses $qA + (1-q)B$ over $pA + (1-p)B$; *continuous* if whenever the player chooses A over B and chooses B over C, there is a probability p for which the player is indifferent between B and the lottery $(1-p)A + pC$; *substitutable* if whenever the player is indifferent between A and B and $0 \leq p \leq 1$, the player is indifferent between $(1-p)A + pC$ and $(1-p)B + pC$.

Theorem

A player has vNM preferences over the lotteries $\mathcal{L}$ associated with a finite set of outcomes and a vNM utility function $u : \mathcal{L} \to \mathbb{R}$ satisfying the Expected Utility Hypothesis if and only if the player's pairwise choices within $\mathcal{L}$ are complete, transitive, generalizable, probabilistic, monotonic, continuous, and substitutable.

Prize Distribution

- Anyone who wants a chance at obtaining one the prizes (chocolate, book, cards, \$10 gift, nothing) should hand me their index card.
- Randomly choose an index card and offer to the chosen participant: (1) item A, or (2) the lottery of receiving B or C with the flip of a coin. The participant should make a choice and explain whether it is consistent with the vNM utility function given on their index card.
- Repeat if there is sufficient time.

Ideas and Practices

- Mathematical modeling process
- Ordinal and vNM utility function construction to model choice
- Axioms to make explicit modeling assumptions
- Utility functions can incorporate what happens to others
- Real agent decisions may not be modeled by a utility function
- Active student involvement
- Index cards for recording and randomization