Game and Group Theories Together

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Coalition Game

A *coalition game* consists of a set N of players and a real-valued worth $w(S)$ for each coalition (nonempty subset) S of players.

S	$w(S)$
$\{1,2,3\}$	60
$\{1,2\}$	48
$\{1,3\}$	24
$\{2,3\}$	18
$\{1\}$	0
$\{2\}$	0
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Coalition Game and Prenucleolus

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The *prenucleolus* is the allocation x that successively maximizes the minimum coalition excesses $e(x, S) = \sum_{i \in S} x_i - w(S)$.

S	$w(S)$	Excess for		
$\{1,2,3\}$	60	$(26, 23, 11)$		
$\{1,2\}$	48	$26 + 23 - 48$	$=$	1
$\{1,3\}$	24	$26 + 11 - 24$	$=$	13
$\{2,3\}$	18	$23 + 11 - 18$	$=$	16
$\{1\}$	0	$26 - 0$	$=$	26
$\{2\}$	0	$23 - 0$	$=$	23
$\{3\}$	0	$11 - 0$	$=$	11

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$\{1, 2, 3\}$	60				
$\{1, 2\}$	48	1	$29 + 25 - 48$	$=$	6
$\{1, 3\}$	24	13	$29 + 6 - 24$	$=$	11
$\{2, 3\}$	18	16	$25 + 6 - 18$	$=$	13
$\{1\}$	0	26	$29 - 0$	$=$	29
$\{2\}$	0	23	$25 - 0$	$=$	25
$\{3\}$	0	11	$6 - 0$	$=$	6

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$\{1, 2\}$	48	1	6	$30 + 24 - 48 = 6$
$\{1, 3\}$	24	13	11	$30 + 6 - 24 = 12$
$\{2, 3\}$	18	16	13	$24 + 6 - 18 = 12$
$\{1\}$	0	26	29	$30 - 0 = 30$
$\{2\}$	0	23	25	$24 - 0 = 24$
$\{3\}$	0	11	6	$6 - 0 = 6$

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Thus, $x_1 + x_2 = 54$ and $x_3 = 6$.

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$$54 + 2(6) - 24 - 18 \geq 12 + 12.$$

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Thus, $x_1 = 30$ and $x_2 = 24$.

Group

A *group* is a set G and an operation $*$ satisfying

- Closed: $(\forall x, y \in G)(x * y \in G)$.
- Identity: $(\exists e \in G)(\forall x \in G)(x * e = e * x = x)$.
- Inverses: $(\forall x \in G)(\exists y \in G)(x * y = y * x = 0)$.
- Associative: $(\forall x, y, z \in G)((x * y) * z = x * (y * z))$.

Example 1. $\mathbb{Z}_n = \{0, 1, 2, \ldots, n - 1\}$ with addition modulo n.

Example 2. Symmetries of a regular polygon with composition.

Example 3. Permutations of $\mathbb{Z}_n$ with composition.

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Definition. The *coalition game on the group* $(G, *)$ consists of the set of players G and the worth $w(S)$ being the number of elements in the smallest subgroup containing S.

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Results for All Finite Groups

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Theorem 2. If ν is the prenucleolus for a coalition game on the non-trivial group $(G, *)$, then $\nu_e = 0$ for the identity element e.

Proof. Suppose to the contrary that $\nu_e > 0$.

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Proof. Suppose to the contrary that $\nu_i < 0$ for some $i \in G$. Consider any coalition S not containing i. Then the excess $e(\nu, S) = \sum_{j \in S} \nu_j - w(S)$ $> \sum_{j \in S} \nu_j + \nu_i - w(S \cup \{i\})$ which is the excess $e(\nu, S \cup \{i\})$. By giving a small amount to player i from the other players, the minimum excess is strictly increased, a contradiction to ν being the prenucleolus.

Theorem 2. If ν is the prenucleolus for a coalition game on the non-trivial group $(G, *)$, then $\nu_e = 0$ for the identity element e.

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Another General Result

Theorem 3. If p is prime, then the prenucleolus ν of the coalition game on $\mathbb{Z}_p$ satisfies $\nu_i = \begin{cases} p/(p-1) & i \neq 0 \\ 0 & i = 0 \end{cases}$.

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By transferring a small amount from player j to player i, the minimum excess will be increased, a contradiction to ν being the prenucleolus. Thus, $\nu_i = \nu_j$ for any two players $i, j \in \mathbb{Z}_n$.

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Conjecture. Suppose n is the product of 2 or more primes $p_1 < \cdots < p_m$. The prenucleolus ν of the coalition game on $\mathbb{Z}_n$ satisfies

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Proof. Let

- A be the set of generators,
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Suppose x is an allocation that maximizes the minimum excess.

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Inequalities

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\begin{array}{ll}
x_i \geq \lambda, & i \in A \\
x_j + x_k \geq \lambda, & j \in B, k \in C_0 \\
x_i \geq 0, & k \in C_0 \\
x_i \geq 0, & k \in C - C_0
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Inequalities		A	B	C_0	$C - C_0$	number
$x_i \geq \lambda,$	$i \in A$	1	0	0	0	$\lvert A\rvert$
$x_j + x_k \geq \lambda,$	$j \in B, k \in C_0$	0	$\lvert C_0\rvert$	$\lvert B\rvert$	0	$\lvert B\rvert\lvert C_0\rvert$
$x_i \geq 0,$	$k \in C_0$	0	0	1	0	$\lvert C_0\rvert$
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$x_j + x_k \geq \lambda,$	$j \in B, k \in C_0$	0	$\lvert C_0\rvert$	$\lvert B\rvert$	0	$\lvert B\rvert\lvert C_0\rvert$	$1/\lvert C_0\rvert$
$x_i \geq 0,$	$k \in C_0$	0	0	1	0	$\lvert C_0\rvert$	$1 - \lvert B\rvert/\lvert C_0\rvert$
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Multiplying each inequality of each type by the corresponding weight and summing, we obtain $n = \sum_{i=0}^{n} x_i \geq (|A| + |B|)\lambda = n$.

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$x_i \geq \lambda,$	$i \in A$	1	0	0	0	$\|A\|$	1
$x_j + x_k \geq \lambda,$	$j \in B, k \in C_0$	0	$\|C_0\|$	$\|B\|$	0	$\|B\|\|C_0\|$	$1/\|C_0\|$
$x_i \geq 0,$	$k \in C_0$	0	0	1	0	$\|C_0\|$	$1 - \|B\|/\|C_0\|$
$x_i \geq 0,$	$k \in C - C_0$	0	0	0	1	$\|C - C_0\|$	1

Multiplying each inequality of each type by the corresponding weight and summing, we obtain $n = \sum_{i=0}^{n} x_i \geq (|A| + |B|)\lambda = n$. All of the inequalities must hold with equality. Hence, $x_i = 0$ for all $i \in C$,

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$x_i \geq 0,$	$k \in C_0$	0	0	1	0	$\lvert C_0\rvert$	$1 - \lvert B\rvert/\lvert C_0\rvert$
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This works as long as $1 - |B|/|C_0| > 0$ or $|C_0| > |B|$.

The End of Summer Conjecture

For a group $\mathbb{Z}_n$ with addition modulo n, let

- A be the set of generators,
- C be the largest proper subgroup of G generated by the smallest prime factor p_1 of $|G|$, and
- B be the set of all the elements not in A or C.

The prenucleolus ν satisfies

Condition	$\nu_i, i \in A$	$\nu_j, j \in B$	$\nu_k, k \in C - \{0\}$	nu_0
$\lvert B\rvert < \lvert C\rvert$	λ	λ	0	0
$\lvert B\rvert > \lvert C\rvert$	λ	$\lambda/2$	$\lambda/2$	0
$\lvert B\rvert = \lvert C\rvert$		——— Impossible ———		

where λ is chosen so that ν is an allocation.

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- Define the coalition game differently.
- Is there some direction that will provide new insights into group theory or game theory?