Congressional Apportionment

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The Constitutional Basis

"Representatives and direct Taxes shall be apportioned among the several States which may be included within this Union, according to their respective Numbers . . . . The actual Enumeration shall be made within three years after the first meeting of the Congress of the United States, and within every subsequent Term of ten Years, in such manner as they shall by Law direct."
article I, section 2

What is the Problem?

Consider populations from the 1990 census ...

$$\text{quota}_{\text{CA}} = \frac{\text{population of CA}}{\text{population of USA}} \times \text{house size} = \frac{33,930,798}{281,434,177} \times 435 = 52.447$$

$$\text{quota}_{\text{UT}} = \frac{\text{population of UT}}{\text{population of USA}} \times \text{house size} = \frac{2,236,714}{281,434,177} \times 435 = 3.457$$

The official apportionment was

$$\text{apportionment}_{\text{CA}} = 53$$
$$\text{apportionment}_{\text{UT}} = 3$$

United States of Arithmetic

State i	Population p_i	Quota q_i	Apportionment a_i
Add	9,598	47.99	
Sub	5,868	29.34	
Mul	2,664	13.32	
Div	1,870	9.35	
Total	20,000	100.00	100

- Simple rounding does not work.
- If we start by rounding down, how should we distribute the two remaining seats?
  - In order of populations p_i (ascending or descending)?
  - In order of remainders $r_i = q_i - \lfloor q_i \rfloor$ (ascending or descending)?
  - In order of relative remainders r_i / p_i (ascending or descending)?

Hamilton's Method

Give to each state the whole number contained in its quota, and then assign remaining seats to states with the largest quota remainders.

State i	Population p_i	Quota q_i	Apportionment a_i
Add	9,598	47.99	$47 + 1 = 48$
Sub	5,868	29.34	$29 + 0 = 29$
Mul	2,664	13.32	$13 + 0 = 13$
Div	1,870	9.35	$9 + 1 = 10$
Total	20,000	100.00	100

Jefferson's Method

Choose an ideal district size. Compute the ratios of population to the ideal district size. Give each state the whole number in its ratio seats. If the house size is fixed, the ideal district size must be chosen so that the seats assigned matches the house size.

State i	Population p_i	Ratio $p_i/200$	Apportionment a_i
Add	9,598	47.99	47
Sub	5,868	29.34	29
Mul	2,664	13.32	13
Div	1,870	9.35	9
Total	20,000		98

Jefferson's Method

Choose an ideal district size. Compute the ratios of population to the ideal district size. Give each state the whole number in its ratio seats. If the house size is fixed, the ideal district size must be chosen so that the seats assigned matches the house size.

State	Population	Ratio	Apportionment
i	p_i	$p_i/195.7$	a_i
Add	9,598	49.04	49
Sub	5,868	29.98	29
Mul	2,664	13.61	13
Div	1,870	9.56	9
Total	20,000		100

Webster's Method

Choose an ideal district size. Compute the ratios of population to the ideal district size. Give each state its **rounded** ratio seats. If the house size is fixed, the ideal district size must be chosen so that the seats assigned matches the house size.

State i	Population p_i	Ratio $p_i/198$	Apportionment a_i
Add	9,598	48.47	48
Sub	5,868	29.63	30
Mul	2,664	13.45	13
Div	1,870	9.44	9
Total	20,000		100

Hill's Method

Choose the apportionment that minimizes the relative difference in average representation between pairs of states.

State i	Population p_i	Quota q_i	Apportionments a_i	a_i
Sub	5,868	29.58	29	30
Div	1,870	9.42	10	9
Total	7,738	39.00	39	39
	Pairwise Measure of Inequity		$\dfrac{\frac{10}{1870} - \frac{29}{5868}}{\frac{10}{1870}} = 0.0758$	$\dfrac{\frac{30}{5868} - \frac{9}{1870}}{\frac{30}{5868}} = 0.0586$

For our example, Hill's and Webster's methods yield the same apportionment. For some distributions of population, the two methods give different results.

Divisor Methods

Choose an ideal district size λ. State i receives p_i/λ rounded with respect to a divisor criterion seats.

OR

Choose an apportionment that minimizes a pairwise measure of inequity.

Method	Divisor	Inequity Measure
Jefferson	$a+1$	$a_i(p_j/p_i) - a_j$
Webster	$a+1/2$	$a_a/p_i - a_j/p_j$
Hill	$\sqrt{a(a+1)}$	$\frac{a_i/p_i}{a_j/p_j} - 1$
Dean	$\frac{a(a+a)}{a+1/2}$	$p_j/a_j - p_i/a_i$
Adams	a	$a_i - a_j(p_i/p_j)$

Does it Make a Real Difference?

For the 1990 Census

State	Quota	Hamilton	Webster	Hill
Massachusetts	10.552	11	11	10
Oklahoma	5.516	5	5	6
New Jersey	13.536	14	13	13
Mississippi	4.518	4	5	5

Jefferson would have changed 16 state apportionments.

For the 2000 Census

Webster is the same as Hill. Hamilton takes a seat from California and gives it to Utah. Jefferson adds two seats to California among several other changes.

For the 2010 Census

Hamilton is the same as Hill. Webster takes a seat from Rhode Island and gives it to North Carolina. Jefferson adds two seats to California among several other changes.

What Method is Best?

"Since the world began there has been but one way of proportioning numbers, namely,

[insert your favorite method here]

nor can there be any other method. This process is purely arithmetical,... If a hundred men were being torn limb from limb, or a thousand babes were being crushed, this process would have no more feeling in the matter than would an iceberg; because the science of mathematics has no more bowels of mercy than has a cast-iron dog."

Representative John A. Anderson of Kansas
Congressional Record 1882, 12:1179

What Method is Best?

- Method definitions are *ad hoc*.
- Webster was used seven times (1840, 1850, 1880, 1890, 1900, 1910, and 1930); Jefferson was used five times (1790 through 1830); Hill was used eight times (1940 through 2010). Twice (1860 and 1870) no consistent method was used; and once (1920) there was no reapportionment.
- Edward V. Huntington (The Apportionment of Representatives in Congress, *Transactions of the American Mathematical Society*, 1928) made the first systematic study of methods based upon measures of inequity..
- Michel L. Balinski and H. Peyton Young (*Fair Representation: Meeting the Ideal of One Man, One Vote*, 1982) use an axiomatic approach based upon desirable properties.

Fair Share

The number of seats assigned a state should be its quota rounded down or up.

State i	Population p_i	Quota q_i	Jefferson a_i
Add	9,598	47.99	49
Sub	5,868	29.34	29
Mul	2,664	13.32	13
Div	1,870	9.35	9
Total	20,000		100

- Jefferson's method does **not** satisfy fair share (consider Add in USA).
- **No** divisor method satisfies fair share.
- Hamilton's method **satisfies** fair share.

House Monotonicity

No state loses a seat when the house size increases (populations unchanged).

State	100 seats		100 seats	
i	q_i	a_i	q_i	a_i
Add	47.99	48	48.47	49
Sub	29.34	29	29.63	30
Mul	13.32	13	13.45	13
Div	9.35	10	9.44	9
Total	100.00	100	101.00	101

- Hamilton's method does **not** satisfy house monotonicity (consider Div in USA).
- **All** divisor methods satisfy house monotonicity.
- There are methods satisfying both fair share and house monotonicity.

Population Monotonicity

No state that increases its population should lose a seat to another state that decreases its population (house size unchanged).

State	First Census			Second Census		
i	p_i	q_i	a_i	p_i	q_i	a_i
Add	9,598	47.99	48	9,550	47.99	48
Sub	5,868	29.34	29	5,865	29.47	30
Mul	2,664	13.32	13	2,610	13.12	13
Div	1,870	9.35	10	1,875	9.42	9
Total	20,000	100.00	100	19,900	100.00	100

- Hamilton's method does **not** satisfy population monotonicity (consider Div and Sub of USA).
- **All** divisor methods satisfy population monotonicity.
- There is **no** method satisfying both fair share and population monotonicity.
- Would weaker forms of these properties or other properties characterize methods?

Relative Population Monotonicity

No state that increases its **relative** population should lose a seat to another state that decreases its **relative** population (house size unchanged).

State	First Census			Second Census		
i	p_i	q_i	a_i	p_i	q_i	a_i
Add	9,598	47.99	48	9,550	47.99	48
Sub	5,868	29.34	29	5,865	29.47	30
Mul	2,664	13.32	13	2,610	13.12	13
Div	1,870	9.35	10	1,875	9.42	9
Total	20,000	100.00	100	19,900	100.00	100

- Hamilton's method **does** satisfy relative population monotonicity (notice that Div and Sub both increase their relative populations as can be seen in their quotas).
- Since population monotonicity implies relative population monotonicity, all divisor methods satisfy population monotonicity.

Near Fair Share

The transfer of a seat from one state to another does not simultaneously take both states closer to their quota.

State i	Population p_i	Quota q_i	First a_i	Second a_i
Add	9,598	47.99	47	48
Sub	5,868	29.34	30	29
Mul	2,664	13.32	13	13
Div	1,870	9.35	10	10
Total	20,000	100.00	100	100

- The example shows that the first apportionment is not fair share.
- Hamilton's method satisfies near fair share.
- Websters's method is the unique method satisfying near fair share and population monotonicity.
- Although sounding related, near fair share is independent of fair share.

Unbiased

The probability that state i is favored over state j ($a_i/p_i > a_j/p_j$) equals the probability that state j is favored over state i ($a_j/p_j > a_i/p_i$).

Quota	Jefferson	Webster	Hill	Dean	Adams
9.988	**11**	10	10	10	10
9.064	9	9	9	9	9
7.182	7	**8**	7	7	7
5.260	5	5	**6**	5	5
3.321	3	3	3	**4**	3
1.185	1	1	1	1	**2**

- There is a clear ordering in the five traditional divisor methods from bias towards large states (Jefferson) and bias towards small states (Adams).
- Under a variety of reasonable assumptions about the population probability distribution, Hamilton's method is unbiased and Webster's method is the unique unbiased and proportional divisor method.

Summary

Property	Hamilton	Webster	Hill	Jefferson
Fair Share	Yes	No	No	No
Near Fair Share	Yes	Yes	No	No
Unbiased	Yes	Yes	No	No
Population Monotone	No	Yes	Yes	Yes
Relative Population Monotone	Yes	Yes	Yes	Yes
House Monotone	No	Yes	Yes	Yes

Conclusions

- Webster's or Hamilton's method would be an improvement upon Hill's method.
- Can Hamiliton's method be characterized with fair share, unbiased, and relative population monotonicity?