LAB 5.1 Hard and Soft Springs

In this lab, we continue our study of second-order equations by considering "nonlinear springs." In Sections 2.1 and 2.3, we developed the model of a spring based on Hooke's law. Hooke's law asserts that the restoring force of a spring is proportional to its displacement, and this assumption leads to the second-order equation

$$m\frac{d^2y}{dt^2} + ky = 0.$$

Since the resulting differential equation is linear, we say that the spring is linear. In this case the restoring force is $-ky$. In addition, we assume that the friction or damping force is proportional to the velocity. The resulting second-order equation is

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = 0.$$

Hooke's law is an idealized model that works well for small oscillations. In fact the restoring force of a spring is roughly linear if the displacement of the spring from its equilibrium position is small, but it is generally more accurate to model the restoring force by a cubic of the form $-ky + ay^3$, where a is small relative to k. If a is negative, the spring is said to be hard, and if a is positive, the spring is soft. In this lab we consider the behavior of hard and soft springs for particular values of the parameters. (Your instructor will tell you which parameter value(s) from Table 5.1 to use.)

In your report, you should analyze the phase planes and $y(t)$- and $v(t)$-graphs to describe the long-term behavior of the solutions to the equations:

1. (Hard spring with no damping) The first equation that you should study is the hard spring with no damping; that is, $b = 0$ and $a = a_1$. Examine solutions using both their graphs and the phase plane. Consider the periods of the periodic solutions that have the initial condition $v(0) = 0$. Sketch the graph of the period as a function of the initial condition $y(0)$. Is there a minimum period? A maximum period? If so, how do you interpret these extrema?

2. (Hard spring with damping) Now use the given value of b and $a = a_1$ to introduce damping into the discussion. What happens to the long-term behavior of solutions in this case? Determine the value of the damping parameter that separates the underdamped case from the overdamped case.

3. (Soft spring with no damping) Consider the soft spring that corresponds to the positive value a_2 of a. Over what range of y-values is this model reasonable? Consider the periods of the periodic solutions that have the initial condition $v(0) = 0$. Sketch the graph of the period as a function of the initial condition $y(0)$. Is there a minimum period? A maximum period? Use the phase portrait to help justify your answer.

4. (Soft spring with damping) Using the given values of b and $a = a_2$, what happens to the long-term behavior of solutions in this case? Determine the value of the damping parameter that separates the underdamped case from the overdamped case.

5. From a physical point of view, what's the difference between a hard spring and a soft spring?

Your report: Address each of the five items in the form of a short essay. You may illustrate your essay with phase portraits and graphs of solutions. However, your essay should be complete and understandable without the pictures. Make sure you relate the behavior of the solutions to the motion of the associated mass and spring systems.

Table 5.1
Choices for the parameter values. Assume the mass $m = 1$ unless you are told otherwise by your instructor.

Choice	k	b	a_1	a_2
1	0.1	0.15	−0.005	0.005
2	0.2	0.20	−0.008	0.008
3	0.3	0.20	−0.009	0.009
4	0.2	0.20	−0.005	0.005
5	0.1	0.10	−0.005	0.005
6	0.3	0.20	−0.007	0.007
7	0.3	0.15	−0.007	0.007
8	0.1	0.15	−0.004	0.004
9	0.2	0.15	−0.005	0.005
10	0.3	0.20	−0.008	0.008

LAB 5.2 Higher Order Approximations of the Pendulum

In previous chapters, we studied the behavior of second-order, homogeneous linear equations (like the harmonic oscillator) by reducing them to first-order linear systems. This "reduction" technique can be applied to nonlinear equations as well, and in this lab we study the ideal pendulum and approximations to the pendulum using this technique.

In the text we modeled the ideal pendulum by the second-order, nonlinear equation

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta = 0,$$

where θ is the angle from the vertical, g is the gravitational constant ($g = 32$ ft/s^2), and l is the length of the rod of the pendulum, that is, the radius of the circle on which the mass travels. In this lab we compare the results of numerical simulation of this model with the results obtained from two approximations to this model. The first approximation is a linear approximation given by

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\theta = 0.$$

The second approximation is a cubic approximation

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\left(\theta - \frac{\theta^3}{6}\right) = 0.$$

Recall from calculus that the expression $\theta - \theta^3/6$ represents the first two terms in the power series expansion of $\sin\theta$ about $\theta = 0$. We are especially interested in how close the solutions of the approximations of the ideal pendulum equation are to the original ideal pendulum equation. In particular, we are interested in how closely the periods of the periodic orbits of the approximations of the pendulum equation relate to the periods of the periodic orbits of original equation. Your instructor will tell you what value of the parameter l (the length of the pendulum arm) you should use. Your report should include:

1. A phase portrait analysis for all three equations. Compare and contrast these phase portraits from the point of view of how well the linear and cubic equations approximate the ideal pendulum.

2. In order to study how the periods of the periodic orbits are related, consider the one-parameter family of initial conditions parameterized by θ_0, where $\theta(0) = \theta_0$ and $\theta'(0) = 0$ (no initial velocity). In other words, you should study the various solutions that begin at a given angle with zero velocity. For what intervals of initial conditions do the periods of the periodic orbits of

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\theta = 0$$

   and

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\left(\theta - \frac{\theta^3}{6}\right) = 0$$

   closely approximate the periods of the periodic orbits of the ideal pendulum? (The computation of the periods in the linear approximation can be done exactly using the techniques of Chapter 3. Analytic techniques exist for computing the periods of the periodic orbits of the other two equations, but in this lab you should work numerically.) You should plot graphs of the period as a function of θ_0 using a relatively small table (5, 10, or 15 entries) of periods obtained using direct numerical simulation of the model.

3. Another family of initial conditions is $\theta(0) = 0$ and $\theta'(0) = v_0$. In this family, the initial velocity is the parameter. Initially the pendulum points straight down with a given velocity v_0. What changes from your results in Part 2 above?

4. Suppose you are a clockmaker who makes clocks based on the motion of a pendulum. For each of the three equations, what would you do to double the period of the oscillation?

Your report: Address each of the items above in the form of a short essay. Be as systematic as possible when collecting data, and present this data in a concise and clear format. You may illustrate your essay with phase portraits and graphs of solutions or of the data that you collect. However, your essay should be complete and understandable without the pictures.

LAB 5.3 A Family of Predator-Prey Equations

In this laboratory exercise, you will study a one-parameter family of nonlinear, first-order systems consisting of predator-prey equations. The family is

$$\frac{dx}{dt} = 9x - \alpha x^2 - 3xy$$
$$\frac{dy}{dt} = -2y + xy,$$

where $\alpha \geq 0$ is a parameter. In other words, for different values of α we have different systems. The variable x is the population (in some scaled units) of prey, and y is the population of predators. For a given value of α, we want to understand what happens to these populations as $t \to \infty$.

You should investigate the phase portraits of these equations for various values of α in the interval $0 \leq \alpha \leq 5$. To get started, you might want to try $\alpha = 0, 1, 2, 3, 4$, and 5. Think about what the phase portrait means in terms of the evolution of the x and y populations. Where are the equilibrium points? What does linearization tell you about their types? What happens to a typical solution curve? Also, consider the behavior of the special solutions where either $x = 0$ or $y = 0$ (solution curves lying on the x- or y-axes).

Determine the bifurcation values of α—that is, the values of α where nearby α's lead to "different" behaviors in the phase portrait. For example, $\alpha = 0$ is a bifurcation value because for $\alpha = 0$, the long-term behavior of the populations is dramatically different than the long-term behavior of the populations if α is slightly positive. The technique of linearization suggests bifurcation values.

Your report: After you have determined all of the bifurcation values for α in the interval $0 \leq \alpha \leq 5$, study enough specific values of α to be able to discuss all of the various population evolution scenarios for these systems. In your report, you should describe these scenarios using the phase portraits and $x(t)$- and $y(t)$-graphs. Your report should include:

1. A brief discussion of the significance of the various terms in the system. For example, what does the $9x$ represent? What does the $3xy$ term represent?
2. A discussion of all bifurcations including the bifurcation at $\alpha = 0$. For example, a bifurcation occurs between $\alpha = 3$ and $\alpha = 5$. What does this bifurcation mean for the predator population?

Address the questions above in the form of a short essay, and support your assertions with selected illustrations. (Please remember that although one good illustration may be worth 1000 words, 1000 illustrations are usually worth nothing.)

LAB 5.4 The Glider

Consider a glider flying in the xy-plane (see Figure 5.66). Let $s(t)$ be the speed of the glider along its path at time t and $\theta(t)$ be the angle of the velocity vector of the path with the x-direction at time t. Note that, since the body of the glider points in the direction of motion, θ is also the angle between the glider and the x-direction.

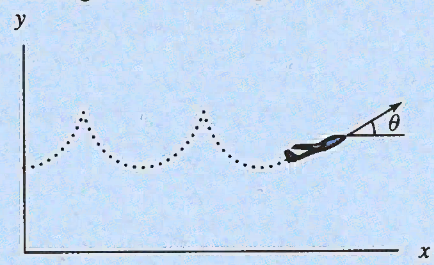

Figure 5.66
The angle θ for the motion of a glider.

The forces involved are gravity, lift provided by the wings (a force perpendicular to the velocity vector), and drag (a force parallel but in the opposite direction to the velocity vector). Using $F = ma$, we can obtain equations for d^2x/dt^2 and d^2y/dt^2 and then derive the system

$$\frac{d\theta}{dt} = \frac{s^2 - \cos\theta}{s}$$

$$\frac{ds}{dt} = -\sin\theta - Ds^2.$$

(The derivation involves several changes of coordinates, including a change of time scale, and is an excellent exercise for your friends who are studying classical mechanics.) Note that this model assumes that both the lift and drag are proportional to the square of the velocity.*

The most remarkable thing about this system is that there is only one parameter, D. This parameter is the drag force caused by air resistance divided by the lift, the "drag-to-lift ratio." The designer of a glider tries to maximize lift while minimizing drag, so the parameter D can be viewed as a measure of the quality of the design (a small value for D is preferred). In this lab we consider the solutions of this model and their relationship to the flight of the glider.

Your report should address the following items:

1. Study the solutions of the system above with $D = 0$ (that is, for the perfect glider with zero drag). Are there equilibrium points? What is the physical interpretation

*This model appears in the book *Theory of Oscillators* by A. A. Andronov, A. A. Vitt, and S. E. Khaikin, Dover Publishing, 1987. Other excellent sources are *Nonlinear Dynamics and Chaos*, by S. Strogatz, Perseus Press, 1994, and *Interactive Differential Equations Workbook* by B. West, S. Strogatz, J. M. Dill, and J. Cantwell, Addison Wesley Interactive, 1997.

(in terms of the path of the glider) of the equilibria? How do the initial conditions relate to the launch of the glider and how does the flight path change with different initial conditions? Show that the quantity $C(\theta, s) = s^3 - 3s\cos\theta$ is a conserved quantity for this system. What does the function C tell you about the solution curves of the system? Are there periodic solutions?

2. Repeat your analysis for values of D between 0 and 4 (that is, for increasing drag-to-lift ratio). How do the phase portraits change as D changes? How do the possible glider paths change as D increases?

3. Apply the theory of linearization to classify all equilibria for values of D in the interval $0 \leq D \leq 4$. Determine the bifurcation values of D.

4. Reconstructing the motion of the glider from the equations: Given a value of D and an initial condition (θ_0, s_0), the motion of the glider is determined from the equations. Show how one can start with values of D and (θ_0, s_0) and obtain the path of the glider. Be precise. (One good way to do this part of the project is to write the code that you would need to draw the path of the glider in some convenient programming language.)

5. Why is it more natural to think of the "phase cylinder" for this system rather than the phase plane? What changes if you analyze the system using a phase cylinder in place of a phase plane?

6. Construct a paper glider and relate test flights to your answers in Parts 1, 2 and 3. (Note: Gliders made from higher-quality paper demonstrate the dynamics much better. You may wish to consult the paper airplane literature for design suggestions. For example, J. M. Collins, *The Gliding Flight*, Ten Speed Press, Berkeley, 1989.)

In your report, pay particular attention to the relationship between the geometry of solutions in the phase plane and the motion of the glider.