Solutions for Partially Defined Coalition Games

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Coalition Games

Definition

Definition (Von Neumann & Morgenstern, 1944). A *coalition game* is a set of players $N = \{1, 2, \ldots, n\}$ and a worth function w from coalitions (nonempty subsets of players) to real numbers.

Example. $N = \{1, 2, 3\}$ and w is defined by the table below.

S	{1,2,3}	{1,2}	{1,3}	{2,3}	{1}	{2}	{3}
w(S)	20	14	8	6	2	0	0

Definition. A coalition game (N, w) is *zero-monotonic* if $w(S) \geq w(S \setminus \{i\}) + w(\{i\})$ for all $i \in S \subset N$.

Shapley Value

Definition. An *allocation method* or *solution* is a function ϕ from coalition games to allocations (n-vectors of reals).

Definition. The Shapley value is an allocation method that gives the ith player her marginal contribution averaged over all possible orders of the players.

Example.

	Marginal Contribution		
Order	Player 1	Player 2	Player 3
123	2	12	6
132	2	12	6
213	14	0	6
231	14	0	6
312	8	12	0
321	14	6	0
Average	9	7	4

Fairness Properties

Definitions. The allocation method ϕ is

- *efficient* if $\sum_{i \in N} \phi_i(N, w) = w(N)$ for all coaltion games (N, w). “All of the potential savings are allocated.”
- *symmetric* if $\phi_{\pi(i)}(N, \pi \circ w) = \phi_i(N, w)$ for all coaltion games (N, w), permutations π of N, and players $i \in N$, where the worth function $\pi \circ w$ is defined by $(\pi \circ w)(\pi(S)) = w(S)$ for all coalitions S. “A player’s name is irrelevant.”
- *dummy subsidy-free* if $\phi_i(N, w) = 0$ for all coaltion games (N, w) and players $i \in N$ satisfying $w(S) = w(S \setminus \{i\})$ for all coalitions S. "Players who never contribute to or detract from the worth of any coalition receive nothing."

- *additive* if $\phi(N, v+w) = \phi(N, v) + \phi(N, w)$ for all coaltion games (N, v) and (N, w). "Accounting procedures are irrelevant."

Characterization Theorem

Theorem (Shapley, 1953). The Shapley value is the unique allocation method that is efficient, symmetric, dummy subsidy-free, and additive (on zero-monotonic coalition games).

Example.

S	{1,2,3}	{1,2}	{1,3}	{2,3}	{1}		ϕ_1	ϕ_2	ϕ_3
v(S)	12	6	6	6	0		4	4	4
$6\,u^{12}(S)$	6	6	0	0	0		3	3	0
$2\,u^{1}(S)$	2	2	2	0	2		2	0	0
w(S)	20	14	8	6	2		9	7	4

Partially Defined Coalition Games

Definition

Definition (Letscher, 1990 & Willson, 1993). A *partially defined coalition game (PDG)* is a set of players $N = \{1, 2, \ldots, n\}$, a collection of coalitions C containing N, and a worth function w from C to real numbers. In the following, we will assume $C = \{S \subset N : |S| \in M\}$ for some $M \subset N$ containing 1 and n.

Example. $N = \{1, 2, 3, 4, 5\}$, $M = \{1, 4, 5\}$, and w is defined by the table below.

S	12345	1234	1235	1245	1345	2345	i
w(S)	600	480	480	360	180	60	0

Extensions

Definition. An extension of the PDG (N, C, w) is a coalition game $(N, \hat{w})$ satisfying $\hat{w}(S) = w(S)$ for all $S \in C$. Let ext(w) denote the set of all zero-monotonic extensions of the PDG w, where N and C are clear from context.

Example. $N = \{1, 2, 3, 4, 5\}$ and $\hat{w}$ is defined by the tables below.

S	12345	1234	1235	1245	1345	2345	i
ŵ(S)	600	480	480	360	180	60	0

S	123	124	125	134	135	145	234	235	245	345
ŵ(S)	300	240	240	120	120	120	40	40	40	40

S	12	13	14	15	23	24	25	34	35	45
ŵ(S)	120	60	60	60	20	20	20	20	20	20

Definition. The PDG (N, C, w) is *zero-monotonic* if it has a zero-monotonic extension.

Normalized Shapley Value

Definition. An *allocation method* is a function ϕ from PDGs to allocations (n-vectors of reals).

Definition. The *normalized Shapley value* for the PDG w is the Shapley value on the extension $\hat{w}$ in which $\hat{w}(S) = b_{|S|}$ if $|S| \in N \setminus M$ for arbitrary real numbers b_k, $k \in N \setminus M$.

Fairness Properties

Definitions. The allocation method ϕ is

- *efficient* if $\sum_{i \in N} \phi_i(N, C, w) = w(N)$ for all PDGs (N, C, w). "All of the potential savings are allocated."

- *symmetric* if $\phi_{\pi(i)}(N, \pi \circ C, \pi \circ w) = \phi_i(N, C, w)$ for all PDGs (N, C, w), permutations π of N, and players $i \in N$, where the worth function $\pi \circ w$ is defined by $(\pi \circ w)(\pi(S)) = w(S)$ for all coalitions S. "A player's name is irrelevant."
- *dummy subsidy-free* if $\phi_i(N, C, w) = 0$ for all PDGs (N, C, w) and players $i \in N$ satisfying $\hat{w}(S) = \hat{w}(S \backslash \{i\})$ for all coalitions S and zero-monotonic extensions $(N, \hat{w})$. "Players who never contribute to or detract from the worth of any coalition receive nothing."
- *additive* if $\phi(N, C, v + w) = \phi(N, C, v) + \phi(N, C, w)$ for all PDGs (N, C, v) and (N, C, w). "Accounting procedures are irrelevant."

Characterization Theorem

Theorem (Housman, 2001). Suppose $M = \{1, k, k+1, \ldots, l, n\}$ for some natural numbers k and l satisfying $1 \leq k \leq l \leq n$. The normalized Shapley value is the unique allocation method that is efficient, symmetric, dummy subsidy-free, and additive on zero-monotonic PDGs.

Example.

S	12345	1234	1235	1245	1345	2345		ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5
$u^1(S)$	60	0	0	0	0	60		0	15	15	15	15
$u^2(S)$	180	0	0	0	180	0		45	0	45	45	45
$u^3(S)$	360	0	0	360	0	0		90	90	0	90	90
$u^4(S)$	480	0	480	0	0	0		120	120	120	0	120
$u^5(S)$	480	480	0	0	0	0		120	120	120	120	0
$u^0(S)$	960	0	0	0	0	0		192	192	192	192	192
w(S)	600	480	480	360	180	60		183	153	108	78	78

Different Extensions and Caveats

Theorem (Housman, 2001). Suppose $M = \{1, k, k+1, \ldots, n\}$ for some natural number $k \leq n$. The normalized Shapley value is the unique allocation method that is efficient, symmetric, dummy subsidy-free, and additive on superadditive PDGs.

Theorem (Housman, 2001). Suppose $M = \{1, n\}$. The normalized Shapley value is the unique allocation method that is efficient, symmetric, dummy subsidy-free, and additive on convex PDGs.

Theorem (Housman, 2001). The reduced Shapley value is the unique allocation method that is efficient, symmetric, dummy subsidy-free, and additive on size monotonic PDGs.

Remark. When the set of extensions is a subset of the zero-monotonic extensions that includes the unanimity games, then the uniqueness argument continues to hold; however, existence may fail! Examples include for all of the previous theorems when M is not of the specified form.

Why not use the normalized Shapley value?

Answer 1

Our previous remark shows us that it may not satisfy our fairness properties. Of course, then no method satisfies our fairness properties.

Answer 2

But even if the context is such that the normalized Shapley value is the unique method satisfying our fairness properties, then there is a strong argument against using it.

Example. $N = \{1, 2, 3, 4, 5\}$, $M = \{1, 4, 5\}$, and w is defined by the table below.

S	12345	1234	1235	1245	1345	2345	i
w(S)	5	4	3	2	1	0	0

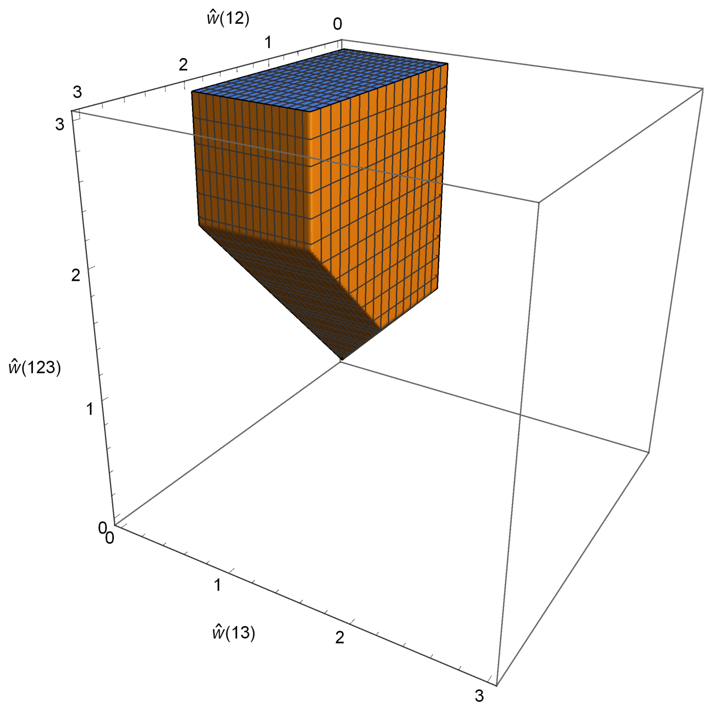
Any zero-monotonic extension satisfies

$$0 \leq \hat{w}(\{i, j\}) \leq \hat{w}(\{i, j, k\}) \leq w(\{i, j, k, l\}) \text{ for all } \{i, j, k, l\} \subset N$$

S	123	124	125	134	135	145	234	235	245	345
$\hat{w}$(S)	≤3	≤2	≤2	≤1	≤1	≤1	0	0	0	0

S	12	13	14	15	23	24	25	34	35	45
$\hat{w}$(S)	0≤	0≤	0≤	0≤	0	0	0	0	0	0

The set of extensions for this PDG is a 10-dimensional convex pyramid-like shape. The graph below captures the constraints on 3 dimensions.



The normalized Shapley value of w equals the Shapley value of the extension $\hat{w}$ for which $\hat{w}(\{i, j\}) = \hat{w}(\{i, j, k\}) = 0$, and this is the only zero monotonic extension of w.

This is the apex of the "pyramid" of extensions.

The normalized Shapley value gives player 1 the <u>minimum</u> payoff, 1.5, among Shapley values of all possible extensions. The maximum payoff among Shapley values of all possible extensions is 2.083.

Conclusion

Additivity is too strong of a property because we ask for the allocations to add even when the sets of extensions do not.

Weakly Additive Allocation Methods

Fairness Properties

Definitions. The allocation method ϕ is

- *weakly additive* if $\phi(N, C, v + w) = \phi(N, C, v) + \phi(N, C, w)$ whenever $\text{ext}(N, C, v + w) = \text{ext}(N, C, v) + \text{ext}(N, C, w)$.
- *proportional* if $\phi(N, C, a\, w) = a\, \phi(N, C, w)$ for all real numbers a and PDGs (N, C, w).

Extension Additivity

Assumption. For the rest of the results, assume that $M = \{1, n - 1, n\}$ and all extensions are zero-monotonic.

Lemma 1. $\text{ext}(v + w) = \text{ext}(v) + \text{ext}(w)$ if and only if the coalitional worths for the $n - 1$ player coalitions are in the same order.

Important PDGs

Definition. Let v^k be defined by $v^k(N) = v^k(N \setminus \{j\}) = 1$ for $j > k$, and $v^k(S) = 0$ otherwise.

Examples.

S	12345	1234	1235	1245	1345	2345	i
$v^1(S)$	1	1	1	1	1	0	0
$v^2(S)$	1	1	1	1	0	0	0
$v^3(S)$	1	1	1	0	0	0	0
$v^4(S)$	1	1	0	0	0	0	0
$v^5(S)$	1	0	0	0	0	0	0

Lemma 2. If ϕ is efficient and symmetric, then $\phi(v^0)$ and $\phi(v^n)$ are completely determined and $\phi(v^k)$ for each $k \in \{1, 2, \ldots, n - 1\}$ is determined up to a single parameter. Furthermore, if ϕ is also dummy subsidy-free, then $\phi(v^{n-1})$ is completely determined.

Examples.

i	1	2	3	4	5
$\phi_i(v^1)$	a_1	$\frac{1}{4}(1 - a_1)$	$\frac{1}{4}(1 - a_1)$	$\frac{1}{4}(1 - a_1)$	$\frac{1}{4}(1 - a_1)$
$\phi_i(v^2)$	$\frac{1}{2}a_2$	$\frac{1}{2}a_2$	$\frac{1}{3}(1 - a_2)$	$\frac{1}{3}(1 - a_2)$	$\frac{1}{3}(1 - a_2)$
$\phi_i(v^3)$	$\frac{1}{3}a_3$	$\frac{1}{3}a_3$	$\frac{1}{3}a_3$	$\frac{1}{2}(1 - a_3)$	$\frac{1}{2}(1 - a_3)$
$\phi_i(v^4)$	$\frac{1}{4}a_4$	$\frac{1}{4}a_4$	$\frac{1}{4}a_4$	$\frac{1}{4}a_4$	$1 - a_4$
$\phi_i(v^5)$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$

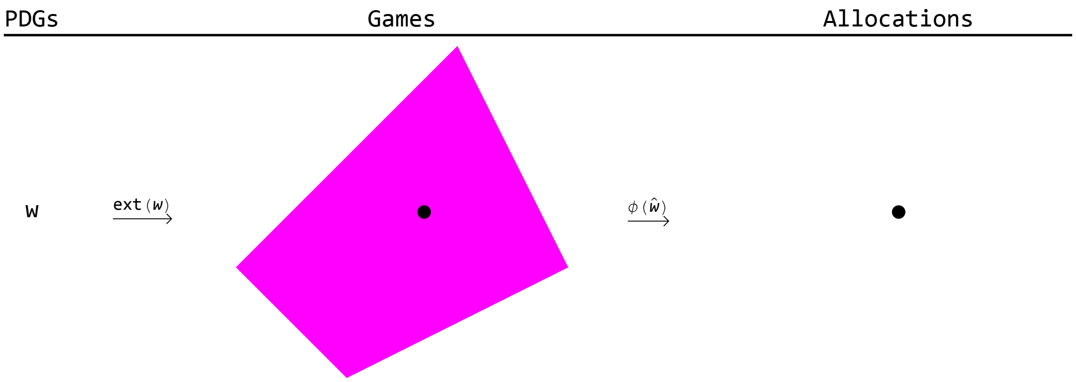
Characterization Theorem

Theorem (Housman, 2004). There exist allocation methods that are efficient, symmetric, dummy subsidy-free, proportional, and weakly additive. They are parameterized by the allocations to player 1 in the PDGs v^1, v^2, ..., v^{n-2}.

Geometric Approach

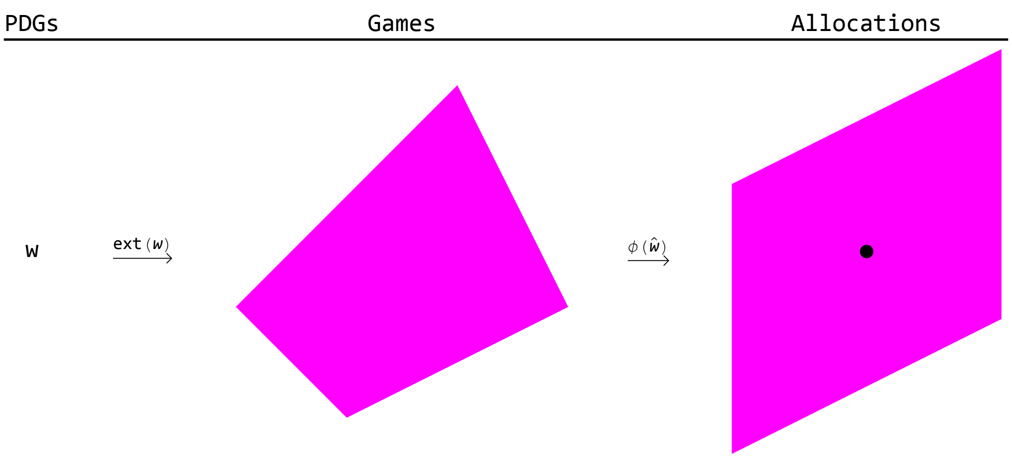
Approach 1

Let $\phi(w)$ be the Shapley value of a central extension of the PDG w.



Approach 2

Let $\phi(w)$ be the central Shapley value of extensions of the PDG w.



As in the axiomatic approach, the allocation depends on the set of extensions (zero-monotonic, superad-ditive, convex). In addition, the geometric approach depends on the definition of center.

Centroid

Difficult.

Coordinate Center

Definition. Let e^T be the coalition game satisfying $e^T(T) = 1$ and $e^T(S) = 0$ for $S \neq T$.

Definition. Suppose w is a PDG. The extension $\hat{w}$ is a *coordinate center* of ext(w) if $\hat{w}$ is the midpoint of the line segment $\{\hat{w} + \lambda e^T : \lambda \in \mathbb{R}\} \cap \text{ext}(w)$ for all coalitions T.

Theorem (Brutt, 1994). For each zero-monotonic PDG, a coordinate center exists and is unique.

Corollary (Housman, 2004). The Shapley value of the coordinate center is an efficient, symmetric, dummy subsidy-free, proportional, and weakly additive allocation method.

Chebyshev Center

Definition. Suppose w is a PDG. The extension $\hat{w}$ is the *Chebyshev center* of ext(w) if $\hat{w}$ is the center of the smallest hypersphere containing ext(w).

Theorem (Engelsone, 1999). For the zero-monotonic PDG w, the Chebyshev center $\hat{w}$ of ext(w) is given by the formula $\hat{w}(S) = \frac{1}{2} \min \{w(T) : S \subset T\}$.

Corollary (Housman, 2004). The Shapley value of the Chebyshev center is an efficient, symmetric, dummy subsidy-free, proportional, and weakly additive allocation method.

Comparison

	Shapley	Centroid	Chebyshev	CoorCent	MaxTo1
$\phi_1(v^1)$	0.400	0.583	0.600	0.600	0.800
$\phi_1(v^2)$	0.350	0.420	0.425	0.433	0.500
$\phi_1(v^3)$	0.300	0.317	0.317	0.317	0.333
$\phi_1(v^4)$	0.250	0.250	0.250	0.250	0.250
$\phi_1(v^5)$	0.200	0.200	0.200	0.200	0.200

S	12345	1234	1235	1245	1345	2345	i
$v^1(S)$	1	1	1	1	1	0	0
$v^2(S)$	1	1	1	1	0	0	0
$v^3(S)$	1	1	1	0	0	0	0
$v^4(S)$	1	1	0	0	0	0	0
$v^5(S)$	1	0	0	0	0	0	0

Related Work

Masuya & Inuiguchi, A fundamental study for partially defined cooperative games, 2015.

Conclusions

The problem of finding a fair allocation method for partially defined cooperative games has not been resolved.

One resolution may come from an appropriate axiomatic characterization.

Another resolution may come from using an allocation method for fully defined games on the center of the extensions of a partially defined game.

These resolutions depend upon

- which coalitional worths are known (e.g., those with cardinalities 1, n - 1, and n),
- what is known about the unknown coalitional worths (e.g., zero-monotonic),
- what definition of center is used (e.g., coordinate center), and
- what axioms are used (e.g., efficient, symmetric, dummy subsidy-free, and weak additive).

There are plenty of questions to be answered!

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